On the stability of normal states for a generalized Ginzburg-Landau model

Ayman Kachmar
Université de Paris-Sud,
Département de mathématiques,
Bât. 425, 91405 Orsay, France
E-mail : ayman.kachmar@math.u-psud.fr

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Abstract

We formulate a spectral problem related to the onset of superconductivity for a generalized Ginzburg-Landau model, where the order parameter and the magnetic potential are defined in the whole space. This model is devoted to the 'proximity effect' for a superconducting sample surrounded by a normal material. In the regime when the Ginzburg-Landau parameter (of the superconducting material) is large, we estimate the critical applied magnetic field for which the normal state will lose its stability, a result that has some roots in the physical literature. In some asymptotic situations, we recover results related to the 'standard' Ginzburg-Landau model, where we mention in particular the two-term expansion for the upper critical field obtained by Helffer-Pan.

Keywords and phrases: generalized Ginzburg-Landau equations, proximity effects, Schrödinger operator with magnetic field, semiclassical analysis.

Contents

1 Introduction and main results

1.1 The generalized Ginzburg-Landau model

It is predicted by the physicist de Gennes [18, 19] that the presence of a normal material exterior to a superconductor will push the superconducting electron Cooper pairs to flow through the normal material and to penetrate significatively over a band of length $\frac{1}{b}$, called the 'extrapolation length'. To understand this phenomenon, which is called by de Gennes the 'proximity effect', one has to consider a generalized Ginzburg-Landau theory where the order parameter and the magnetic potential are both defined in the whole space.
In the Ginzburg-Landau theory (cf. [22]), the superconducting properties are described by a complex valued wave function ψ, called the 'order parameter', and a real vector field $\mathcal{A}$, called the 'magnetic potential'. The pair (ψ, A) has the following physical interpretation : $|\psi|^2$ measures the density of the electron Cooper pairs (in particular, $\psi \equiv 0$ corresponds to a normal state) and $\operatorname{curl} A$ measures the induced magnetic field in the sample. For cylindrical superconductors with infinite height and placed in an applied magnetic field parallel to the axis of the cylinder, it is sufficient to define the pair (ψ, A) on $\mathbb{R}^2$ (i.e. on the 2-D cross section). The system is in equilibrium when the pair (ψ, A) minimizes the 'Gibbs free energy'.
After a proper scaling, the 'Gibbs free energy' has the following form (cf. [12]) :

$$(\psi, A) \mapsto \mathcal{G}(\psi, A) = \int_{\mathbb{R}^2} \left\{ \frac{1}{\widetilde{m}} |\nabla_{\kappa H A}\psi|^2 + \widetilde{a}\kappa^2 |\psi|^2 + \widetilde{\beta}\frac{\kappa^2}{2}|\psi|^4 + (\kappa H)^2 |\operatorname{curl} A - 1|^2 \right\} dx, \tag{1.1}$$

where we use the notation,

$$\nabla_{\kappa H A}\psi = (\nabla - i\kappa H A)\psi. \tag{1.2}$$

The functional (1.1) depends on many parameters : $\kappa > 0$ is a temperature independent parameter called the 'Ginzburg-Landau parameter' (it is a characteristic of the superconducting material), $H > 0$ is the intensity of the constant applied magnetic field, $\widetilde{m}, \widetilde{a}, \widetilde{\beta}$ are functions defined in $\mathbb{R}^2$ and are depending on the material, temperature, etc. Typically, the function $\widetilde{a}$ depends on the temperature T in the following way :

$$\widetilde{a} \approx (T - T_c),$$

where T_c is the critical temperature. As in [12], we take the functions $\widetilde{m}, \widetilde{a}, \widetilde{\beta}$ in the following form :

$$\widetilde{m} = \begin{cases} 1, & \text{in } \Omega \\ m, & \text{in } \mathbb{R}^2 \setminus \Omega \end{cases} \qquad \widetilde{a} = \begin{cases} -1, & \text{in } \Omega \\ a, & \text{in } \mathbb{R}^2 \setminus \Omega \end{cases} \qquad \widetilde{\beta} = \begin{cases} 1, & \text{in } \Omega \\ 0, & \text{in } \mathbb{R}^2 \setminus \Omega. \end{cases} \tag{1.3}$$

Here $\Omega \subset \mathbb{R}^2$ is assumed to be open, bounded and simply connected, and a, m are positive constants. Typically, Ω corresponds to a superconducting material[1] (i.e. below its critical temperature) surrounded by a normal material (i.e. above its critical temperature).
If $(\psi, A) \in H^1_{\rm loc}(\mathbb{R}^2; \mathbb{C}) \times H^1_{\rm loc}(\mathbb{R}^2; \mathbb{R}^2)$ is a critical point of $\mathcal{G}$, then the following condition holds for any $(\phi, B) \in C_0^\infty(\mathbb{R}^2; \mathbb{C}) \times C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(\frac{1}{\widetilde{m}} \Re\left(\nabla_{\kappa H A}\psi \cdot \overline{\nabla_{\kappa H A}\phi}\right) - \frac{\kappa H}{\widetilde{m}} B \cdot \Im\left(\overline{\psi}\nabla_{\kappa H A}\psi\right) + (\kappa H)^2 \operatorname{curl} A \cdot \operatorname{curl} B \right) dx \\ &\quad = (\kappa H)^2 \Re\left(\int_\Omega (1 - |\psi|^2)\psi\overline{\phi}\, dx - a \int_{\Omega^c} \psi\overline{\phi}\, dx \right). \end{aligned}$$

[1] We emphasize here that the Ginzburg-Landau parameter κ is determined *only* by the material in Ω.

Thus, (ψ, A) is a weak-solution of the following system of equations, which we call 'generalized Ginzburg-Landau equations',

$$\begin{cases} -\nabla^2_{\kappa HA}\psi = (\kappa H)^2(1-|\psi|^2)\psi, \quad \text{in } \Omega, \\ \operatorname{curl}^2 A = \frac{1}{\kappa H}\Im\left(\overline{\psi}\nabla_{\kappa HA}\psi\right), \quad \text{in } \Omega, \\ -\frac{1}{m}\nabla^2_{\kappa HA}\psi + a(\kappa H)^2\psi = 0, \quad \text{in } \overline{\Omega}^c, \\ \operatorname{curl}^2 A = \frac{1}{m(\kappa H)}\Im\left(\overline{\psi}\nabla_{\kappa HA}\psi\right), \quad \text{in } \overline{\Omega}^c, \\ \mathcal{T}^{\rm int}_{\partial\Omega}\{\nu\cdot\nabla_{\kappa HA}\psi\} = \frac{1}{m}\mathcal{T}^{\rm ext}_{\partial\Omega}\{\nu\cdot\nabla_{\kappa HA}\psi\}, \quad \mathcal{T}^{\rm int}_{\partial\Omega}\psi = \mathcal{T}^{\rm ext}_{\partial\Omega}\psi, \\ \mathcal{T}^{\rm int}_{\partial\Omega}(\operatorname{curl} A) = \mathcal{T}^{\rm ext}_{\partial\Omega}(\mathrm{curlA}), \quad \mathcal{T}^{\rm int}_{\partial\Omega}(A) = \mathcal{T}^{\rm ext}_{\partial\Omega}(A) \quad \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In the above equations, ν is the unit outward normal vector of $\partial\Omega$. We use the notations $\mathcal{T}^{\rm int}_{\partial\Omega}$ and $\mathcal{T}^{\rm ext}_{\partial\Omega}$ to denote, respectively, the 'interior' and the 'exterior' trace on $\partial\Omega$:

$$\mathcal{T}^{\rm int}_{\partial\Omega}\ : H^1(\Omega) \longrightarrow L^2(\partial\Omega), \quad \mathcal{T}^{\rm ext}_{\partial\Omega}\ : H^1(\Omega^c) \longrightarrow L^2(\partial\Omega).$$

Although an increasing number of mathematicians become interested in the problems arising from superconductivity, very few attention is paid to the functional (1.1). In the former literature, the authors are either concerned with the minimization of an energy functional defined only in Ω (cf. (1.21)), or they replace the energy of the normal material by a boundary term (cf. (1.20)). We mention here for instance the works of Bernoff-Sternberg [7], Baumann-Phillips-Tang [6], Lu-Pan [34], Helffer-Morame [24], Helffer-Pan [25], Fournais-Helffer [17] and Kachmar [31]. Other 'generalized' energy functionals similar to (1.1) and that models 'inhomogeneous' superconducting samples were also analyzed previously, see for instance [2, 4] and the references therein. The inhomogeneity there is only due to variations of the 'critical temperature' within the sample (i.e. $\widetilde{m} = 1$ and $\widetilde{a}$ varies), and the authors were mainly concerned with the analysis of 'pinning effects', that is, roughly speaking, the attraction of vortices towards the less superconducting regions (unlike our situation where we are concerned with the analysis of the onset of superconductivity).
Perhaps it is in [12] that the functional (1.1) is first introduced in the mathematical literature, but the analysis there seems to remain at a formal level. Recently, a rigorous analysis of the functional (1.1) has been carried out by Giorgi [20]. Among other things, the author proves the existence of minimizers for (1.1) in a suitable functional space (related to the analysis of Laplace's equation in $\mathbb{R}^2$, see [3]), the existence of normal states and of an upper critical field. By a normal state we mean a solution of (1.4) of the form $(0, \mathbf{F})$. In our situation, we can choose $\mathbf{F}$ in the following canonical way :

$$\mathbf{F}(x_1, x_2) = \frac{1}{2}(-x_2, x_1), \quad \forall (x_1, x_2) \in \mathbb{R}^2,$$

and we notice that $\mathbf{F}$ satisfies :

$$\operatorname{curl}\mathbf{F} = 1, \quad \operatorname{div}\mathbf{F} = 0, \quad \text{in } \mathbb{R}^2.$$

With the above property, the vector field $\mathbf{F}$ is unique up to a gauge transformation (cf. [20]).

1.2 Statement of the results

A normal state is said to be stable[2] if it is a local minimum of (1.1). The Hessian of (1.1) at the normal state $(0, \mathbf{F})$ is given by :

$$\mathbb{E} \ni (\phi, B) \mapsto 2\mathcal{Q}[\kappa, H](\phi) + 2(\kappa H)^2 \int_{\mathbb{R}^2} |\operatorname{curl} B|^2 dx,$$

where the quadratic form $\mathcal{Q}[\kappa, H]$ is defined by :

$$\mathcal{Q}[\kappa, H](\phi) = \int_\Omega \left(|\nabla_{\kappa H\mathbf{F}}\phi|^2 - \kappa^2|\phi|^2\right) dx + \int_{\Omega^c} \left(\frac{1}{m}|\nabla_{\kappa H\mathbf{F}}\phi|^2 + a\kappa^2|\phi|^2\right) dx. \tag{1.5}$$

Thus, for $(0, \mathbf{F})$ to be stable, a necessary condition is to have :

$$\mathcal{Q}[\kappa, H](\phi) \geq 0, \quad \forall \phi \in \mathcal{H}^1_{\kappa H\mathbf{F}}(\mathbb{R}^2),$$

where, given a vector field $\mathcal{A}$ and an open set $U \subset \mathbb{R}^2$, the space $\mathcal{H}^1_{\mathcal{A}}(U)$ is defined by,

$$\mathcal{H}^1_{\mathcal{A}}(U) = \{u \in L^2(U); \quad \nabla_{\mathcal{A}} u \in L^2(U)\}. \tag{1.6}$$

By Friedrich's Theorem, the quadratic form (1.5) defines a self-adjoint operator[3], the bottom of its spectrum is given by :

$$\mu^{(1)}(\kappa, H) = \inf_{\phi \in \mathcal{H}^1_{\kappa H\mathbf{F}}(\mathbb{R}^2), \phi \neq 0} \left(\frac{\mathcal{Q}[\kappa, H](\phi)}{\|\phi\|^2_{L^2(\mathbb{R}^2)}}\right). \tag{1.7}$$

In terms of (1.7), we define the following 'local' upper critical field,

$$H_{C_3}(a, m; \kappa) = \inf\{H > 0; \quad \mu^{(1)}(\kappa, H) \geq 0\}. \tag{1.8}$$

Below H_{C_3}, normal states will loose their stability. Our aim is to estimate H_{C_3} as $\kappa \to +\infty$. It could be more convenient to use the notation $H^{\mathrm{loc}}_{C_3}$ rather than that in (1.8). However, there are many reasonable definitions of the upper critical field all of which are proved to coincide for a 'standard' model and when the Ginzburg-Landau parameter κ is sufficiently large (cf. [16, 17]). We have chosen the definition of the upper critical field in (1.8) because its analysis is actually purely spectral. We hope to verify in the near future that even for this model, other definitions of the upper critical field will coincide with the definition given in (1.8).

We state now our main results.

Theorem 1.1. *There exists a function* $\alpha :]0, +\infty[\times]0, +\infty[\mapsto]\Theta_0, 1]$ *such that, given* $a, m > 0$*, the upper critical field satisfies,*

$$H_{C_3}(a, m; \kappa) = \frac{\kappa}{\alpha(a, m)}\left(1 + o(1)\right), \quad (\kappa \to +\infty). \tag{1.9}$$

Here $\Theta_0 \in]0, 1[$ *is a universal constant.*

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[2] Our definition of stability is actually that of 'local stability', but in the regime considered in this paper, we expect that 'stability' and 'local stability' of normal states will coincide.

[3] This is a linear elliptic operator with discontinuous coefficients. The theory of such operators is well treated, see [40] for example.

Compared with the former literature ([25, 34, and references therein]), we observe that the value of the upper critical field can be strongly modified. It is well known to physicists that the upper critical field for type II superconductors is strongly dependent on the type of the material placed adjacent to the superconductor (cf. [26]).
To prove Theorem 1.1, we need various estimates on the bottom of the spectrum $\mu^{(1)}(\kappa, H)$ in the regime $\kappa, H \to +\infty$ (cf. (1.7)). Eigenvalue asymptotics for linear elliptic operators with discontinuous coefficients arise in other contexts (cf. [27]), but here the problem is different. We follow the technique of Helffer-Morame [24] by analyzing the model case when $\Omega = \mathbb{R}\times\mathbb{R}_+$ is the half-plane. Actually, let us consider the quadratic form,

$$\mathcal{H}^1_{\kappa H \mathcal{A}_0}(\mathbb{R}^2) \ni \phi \mapsto \mathcal{Q}_{\mathbb{R}\times\mathbb{R}_+}[\kappa, H](\phi), \tag{1.10}$$

where

$$\mathcal{Q}_{\mathbb{R}\times\mathbb{R}_+}[\kappa, H](\phi) = \int_{\mathbb{R}\times\mathbb{R}_+} \left(|(\nabla_{\kappa H A_0}\phi|^2 - \kappa^2|\phi|^2\right) dx + \int_{\mathbb{R}\times\mathbb{R}_-} \left(\frac{1}{m}|\nabla_{\kappa H A_0}\phi|^2 + a\kappa^2|\phi|^2\right) dx,$$

and the magnetic potential A_0 is defined by :

$$A_0(x_1, x_2) = (-x_2, 0), \quad \forall (x_1, x_2) \in \mathbb{R}\times\mathbb{R}. \tag{1.11}$$

We denote by

$$\mu^{(1)}(\kappa, H; \mathbb{R}\times\mathbb{R}_+) = \inf_{\phi\in\mathcal{H}^1_{\kappa H \mathcal{A}_0}(\mathbb{R}^2), \phi\neq 0} \frac{\mathcal{Q}_{\mathbb{R}\times\mathbb{R}_+}[\kappa, H](\phi)}{\|\phi\|^2_{L^2(\mathbb{R}^2)}}. \tag{1.12}$$

Performing the scaling $t = (\kappa H)^{1/2} x_2$ and $z = (\kappa H)^{1/2} x_1$, we get with $\alpha = \kappa/H$,

$$\mu^{(1)}(\kappa, H; \mathbb{R}\times\mathbb{R}_+) = \kappa H\, \beta(a, m, \alpha). \tag{1.13}$$

Here $\beta(a, m, \alpha)$ is the bottom of the spectrum of the self-adjoint operator associated to the quadratic form

$$\mathcal{H}^1_{A_0}(\mathbb{R}^2) \ni v \mapsto \mathcal{Q}[a, m, \alpha](v),$$

which is defined by,

$$\begin{aligned} \mathcal{Q}[a, m, \alpha](v) &= \int_{t>0} \left(|\partial_t v|^2 + |(t - i\partial_z)v|^2 - \alpha|v|^2\right) dt dz \\ &\quad + \int_{t<0} \left(\frac{1}{m}\left[|\partial_t v|^2 + |(t - i\partial_z)v|^2\right] + a\alpha|v|^2\right) d\tau dz. \end{aligned} \tag{1.14}$$

We define $\alpha(a, m)$ as the solution, which will be shown to exist uniquely in Theorem 3.7, of the equation $\beta(a, m, \alpha) = 0$. Notice that this will correspond to the magnetic field $H = \kappa/[\alpha(a, m)]$ that satisfies $\mu^{(1)}(\kappa, H; \mathbb{R}\times\mathbb{R}_+) = 0$.

In the next theorem, we describe the behavior of the function α.

Theorem 1.2. *Given $a > 0$, the function $m \mapsto \alpha(a, m)$ is strictly decreasing, $\alpha(a, m) = 1$ if $m \leq 1$, $\Theta_0 < \alpha(a, m) < 1$ if $m > 1$, and*

$$\lim_{m\to 1^+} \alpha(a, m) = 1. \tag{1.15}$$

Moreover, $\alpha(a, m)$ has the following asymptotic expansion as m tends to ∞ :

$$\alpha(a, m) = \Theta_0 + \frac{3\sqrt{a\Theta_0}C_1}{\sqrt{m}} + \mathcal{O}\left(\frac{1}{m}\right), \tag{1.16}$$

where $C_1 > 0$ is a universal constant.

Let us mention that the universal constants Θ_0 and C_1 are defined via auxiliary spectral problems (cf. (2.13) and (2.14)) and were already present in the analysis of the 'standard model', see however [6, 7, 25, 31, 34] (Θ_0 is indeed the bottom of the spectrum of the Neumann realization of the Schrödinger operator with constant magnetic field in $\mathbb{R}\times\mathbb{R}_+$). Numerically [37, 8], one finds that $\Theta_0 \sim 0.59$. Theorem 1.2 proves the validity regime of the results of [26].

For sufficiently large values of the parameter m, we are able to obtain a two-term asymptotic expansion of the upper critical field, where the scalar curvature plays a major role.

Theorem 1.3. *Given $a>0$, there exist a constant $m_0>1$ and a function*

$$\mathcal{C}_1(a,\cdot):\ [m_0,+\infty[\mapsto \mathbb{R}_+$$

such that, if $m>0$ verifies $m\geq m_0$, then the upper critical field satisfies,

$$H_{C_3}(a,m;\kappa)=\frac{\kappa}{\alpha\,(a,m)}+\frac{\mathcal{C}_1\,(a,m)}{\alpha\,(a,m)^{3/2}}(\kappa_{\mathrm{r}})_{\max}+\mathcal{O}\left(\kappa^{-1/3}\right),\quad \text{as } \kappa\to+\infty, \tag{1.17}$$

where κ_{r} denotes the scalar curvature of $\partial\Omega$.

Let us explain what stands behind the statement of Theorem 1.3. As in [33], we look for a *formal eigenfunction* corresponding to the eigenvalue (1.12) in the form :

$$v(x_1,x_2)=\exp\left(-i\zeta_0(\kappa H)^{1/2}x_1\right)f\left((\kappa H)^{1/2}x_2\right),\quad (x_1,x_2)\in\mathbb{R}\times\mathbb{R}, \tag{1.18}$$

that is, we ask for $\zeta_0\in\mathbb{R}$ and $f\in H^1(\mathbb{R})\cap L^2(\mathbb{R};\tau^2d\tau)$ such that

$$\begin{aligned}\int_{t>0}&\left(|\partial_t f|^2+|(t+\zeta_0)f|^2-\alpha|f|^2\right)d\tau\\ &+\int_{t<0}\left(\frac{1}{m}\left[|\partial_t f|^2+|(t+\zeta_0)f|^2\right]+a\alpha|f|^2\right)d\tau=0.\end{aligned}$$

The existence of ζ_0 occurs only when $m>1$ (cf. Theorem 3.5), and, unlike the case of [24, 33], the uniqueness of ζ_0 is not obvious. We are able to obtain uniqueness of ζ_0 only in the regime $m\to+\infty$, which explains why the conclusion of Theorem 1.3 is limited to sufficiently large values of m. However, we believe that that this restriction is technical and we conjecture that the function $\mathcal{C}_1(a,\cdot)$ can be extended to $]1,m_0]$ so that the asymptotic expansion (1.17) will hold for any $m>1$.

Remark 1.4.

1. *In the regime $m\to+\infty$, Theorems 1.1 and 1.2 give, to a first order approximation, the same behavior as in [34]. This was predicted by the formal computations of [12].*
2. *Let us note that the function $\mathcal{C}_1(\cdot,\cdot)$ satisfies*[4]

$$\lim_{m\to+\infty}\mathcal{C}_1(a,m)=(1+6a\Theta_0^2)C_1,$$

so we recover, for $a=0$ and $m=+\infty$ in (1.17), the two-term asymptotic expansion of Helffer-Pan [25].

[4] This will follow from Proposition 3.6, the expression of $\mathcal{C}_1$ (cf. (7.17) and (4.10)) and the asymptotic behavior as $m\to+\infty$ (cf. Proposition 3.10).

Remark 1.5.

1. *It would be desirable to remove the hypothesis of smoothness of the boundary $\partial\Omega$. As in [9, 11], one can perhaps consider a piecewise smooth domain.*

2. *After having obtained Theorems 1.1-1.3, we have learned about the existence of [21], where the authors deal only with the one-dimensional case.*

1.3 Comparison with the de Gennes model

The physicist de Gennes [18] proposes to model the proximity effect by means of a 'Robin type' boundary condition. He assumes that the order parameter satisfies,

$$\nu\cdot(\nabla - i\kappa HA)\psi + \tilde{\gamma}(\kappa;x)\psi = 0 \quad \text{on } \partial\Omega, \tag{1.19}$$

and that this condition permits one to ignore the behavior of ψ outside Ω. The function $\tilde{\gamma}(\kappa;\cdot)$ is supposed to be smooth and is called (when it is constant) the de Gennes parameter. In this case, one has to replace the energy (1.1) by :

$$H^1(\Omega;\mathbb{C})\times H^1(\Omega;\mathbb{R}^2) \ni (\psi,A) \mapsto \mathcal{E}(\psi,A) = \mathcal{G}_s(\psi,A) + \int_{\partial\Omega}\tilde{\gamma}(\kappa;x)|\psi(x)|^2 d\mu_{|\partial\Omega}(x), \tag{1.20}$$

where $\mathcal{G}_s$ is defined by :

$$\mathcal{G}_s(\psi,A) = \int_\Omega \left\{ |\nabla_{\kappa HA}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + (\kappa H)^2|\operatorname{curl} A - 1|^2 \right\} dx. \tag{1.21}$$

Let us take $\tilde{\gamma}(\kappa;x) = \kappa^\delta\gamma_0$ with $\delta \geq 0$ and $\gamma_0 \in \mathbb{R}$. We can define an upper critical field $H_{C_3}(\delta,\gamma_0;\kappa)$ as before (i.e. below H_{C_3}, normal states will loose their stability). In the next theorem, we give a first order approximation of $H_{C_3}(\delta,\gamma_0;\kappa)$.

Theorem 1.6. *There exists a strictly increasing function $\mathbb{R}\ni\gamma\mapsto\Theta(\gamma)$ such that, as $\kappa\to+\infty$, we have the following asymptotics :*

$$H_{C_3}(\delta,\gamma_0;\kappa) = \begin{cases} \dfrac{\kappa}{\Theta_0}(1+o(1)), & \text{if } 0\leq\delta<1 \text{ or if } \gamma_0=0; \\[2ex] \dfrac{\kappa}{\Theta(\gamma_0\cdot\ell(\gamma_0))}(1+o(1)), & \text{if } \delta=1; \\[2ex] \kappa, & \text{if } \delta>1 \text{ and } \gamma_0>0; \\[2ex] \left(\dfrac{\gamma_0}{\eta_0}\right)^2\kappa^{2\delta-1}, & \text{if } \delta>1 \text{ and } \gamma_0<0. \end{cases} \tag{1.22}$$

Here $\ell(\gamma_0)$ is the unique solution of the implicit equation

$$\Theta(\gamma_0\cdot\ell(\gamma_0)) = \ell(\gamma_0)^2$$

and η_0 is the unique zero of $\Theta(\cdot)$.

For a precise definition of the function $\Theta(\cdot)$, see (2.8). In particular, we have $\Theta(0)=\Theta_0$. The existence of the function $\ell(\cdot)$ is proved in Lemma A.2.

Remark 1.7. *Let us consider the case of Theorem 1.1 when $m > 1$. In this case, $\alpha(a,m) \in]\Theta_0, 1[$ (cf. Theorem 1.2), and there exists a unique $\gamma(a,m) > 0$ such that*

$$\alpha(a,m) = \Theta(\gamma(a,m)).$$

Then by putting,

$$\gamma_0 = \frac{\gamma(a,m)}{\sqrt{\alpha(a,m)}}, \quad \ell(\gamma_0) = \sqrt{\alpha(a,m)},$$

we get,

$$H_{C_3}(a,m;\kappa) = \frac{\kappa}{\Theta(\gamma_0 \cdot \ell(\gamma_0))}(1 + o(1)), \quad as \quad \kappa \to +\infty.$$

Therefore, the result of Theorem 1.1 corresponds to the following boundary condition

$$\nu \cdot (\nabla - i\kappa H A)\psi + \kappa\gamma_0\psi = 0, \qquad on\ \partial\Omega.$$

Remark 1.8. *In Theorem 1.6, if $0 \leq \delta \leq 1$, one can still obtain an asymptotic expansion of $H_{C_3}(\delta, \gamma_0; \kappa)$ involving the scalar curvature (cf. [25, 30]).*

Remark 1.7 suggests that one can replace the spectral problem (1.5)-(1.7) by a suitable problem in Ω (with some de Gennes boundary condition) having the same ground state energies and whose ground states coincide in Ω. However, as the case of the half-plane model will show, this will not be the case and the best one can hope is the convergence of the ground state energies.

Actually, if this were the case for the half-plane, i.e. there exists $\zeta_0 = \zeta_0(a,m) \in \mathbb{R}$ such that the function v given by (1.18) is an eigenfunction of (1.10)-(1.12) and satisfies the boundary condition:

$$\partial_{x_2} v = \gamma_0(a,m)\, v \quad \text{on } \partial\mathbb{R} \times \mathbb{R}_+,$$

then we get that $\zeta_0(a,m) = \xi\big(\gamma(a,m)\big)$, where $\gamma_0(a,m)$ and $\gamma(a,m)$ are given by Remark 1.7, and $\xi(\cdot)$ will be defined in (2.10). On the other hand, by the discussion in Subsection 3.4, and in particular Proposition 3.14, we get that

$$\lim_{m \to +\infty} \sqrt{m}\Big(\zeta_0(a,m) - \xi\big(\gamma(a,m)\big)\Big) > 0,$$

which shows that it is impossible for the ground state to satisfy the boundary condition corresponding to the right value of the ground state energy.

One should also mention in this direction the result for the nonlinear problem obtained recently in [28], where the author proves that for the case without magnetic field, $H = 0$, all minimizers of (1.1) are gauge equivalent to a real phase $(u_\kappa, 0)$ that satisfies as $\kappa \to +\infty$:

$$\nu \cdot \nabla u_\kappa = \kappa\sqrt{\frac{a}{2m}}\, u_\kappa(1 + o(1)) \quad \text{on } \partial\Omega.$$

When compared with Theorems 1.2, 1.6 and Remark 1.7 we get, especially for $m = 1$, that for each regime of the applied magnetic field one should associate a different de Gennes boundary condition. The physical interpretation is that the penetration length, which de Gennes measures in terms of his parameter in the boundary condition, is strongly dependent on the applied magnetic field (as predicted in [15]).

1.4 Organization of the paper

In Section 2, we recall auxiliary material that we shall use frequently in the paper. The analysis of the model problem (1.10) leads in Section 3 to the spectral analysis of a family of ordinary differential operators. We obtain by elementary arguments most of the properties announced in Theorem 1.2, and we complete its proof by a fine asymptotic analysis when the parameter m is large.
In Section 4, we analyze a 'refined' family of model operators, whose study is essential for the proof of Theorem 1.3.
In Section 5, we establish a simpler formula of the upper critical field. Using the results of Section 3, we are able to follow a similar analysis to [24] for estimating the bottom of the spectrum $\mu^{(1)}(\kappa, H)$ and we give a proof of Theorem 1.1.
Since the variational problem (1.7) is over the whole plane $\mathbb{R}^2$, minimizers do not always exist. In Section 6, we establish using Persson's Lemma [35] the existence of minimizers to (1.7) when the intensity of the magnetic field H is near H_{C_3}. We prove also by using the technique of Agmon estimates [1] that the minimizers of (1.7) decay exponentially fast away from the boundary $\partial\Omega$.
We are now ready in Section 7 to imitate the analysis of Helffer-Morame [24, Section 11] and to derive a two term-asymptotic expansion of $\mu^{(1)}(\kappa, H)$ and we use it to prove Theorem 1.3.
Finally, in Appendix A, we give a proof for the asymptotics announced in Theorem 1.6.

2 Auxiliary material

2.1 A family of ordinary differential operators.

The analysis of a canonical operator in the half-plane $\mathbb{R} \times \mathbb{R}_+$ with de Gennes boundary condition leads us naturally to a family of ordinary differential operators (cf. [30]). Given $(\gamma, \xi) \in \mathbb{R} \times \mathbb{R}$, we define the quadratic form,

$$B^1(\mathbb{R}_+) \ni u \mapsto q[\gamma,\xi](u) = \int_{\mathbb{R}_+} \left(|u'(t)|^2 + |(t-\xi)u(t)|^2\right) dt + \gamma |u(0)|^2, \tag{2.1}$$

where, for a positive integer $k \in \mathbb{N}$ and a given interval $I \subseteq \mathbb{R}$, the space $B^k(I)$ is defined by :

$$B^k(I) = \{u \in H^k(I); \quad t^j u(t) \in L^2(I), \quad \forall j = 1, \cdots, k\}. \tag{2.2}$$

By Friedrichs' Theorem, we can associate to the quadratic form (2.1) a self adjoint operator $\mathcal{L}[\gamma, \xi]$ with domain,

$$D(\mathcal{L}[\gamma,\xi]) = \{u \in B^2(\mathbb{R}_+); \quad u'(0) = \gamma u(0)\},$$

and associated to the differential operator,

$$\mathcal{L}[\gamma,\xi] = -\partial_t^2 + (t-\xi)^2. \tag{2.3}$$

We denote by $\{\lambda_j(\gamma,\xi)\}_{j=1}^{+\infty}$ the increasing sequence of eigenvalues of $\mathcal{L}[\gamma,\xi]$. When $\gamma = 0$ we write,

$$\lambda_j^N(\xi) := \lambda_j(0,\xi), \quad \forall j \in \mathbb{N}, \quad \mathcal{L}^N[\xi] := \mathcal{L}[0,\xi]. \tag{2.4}$$

We also denote by $\{\lambda_j^D(\xi)\}_{j=1}^{+\infty}$ the increasing sequence of eigenvalues of the Dirichlet realization of $-\partial_t^2 + (t-\xi)^2$.
By the min-max principle, we have,

$$\lambda_1(\gamma,\xi) = \inf_{u \in B^1(\mathbb{R}_+), u \neq 0} \frac{q[\gamma,\xi](u)}{\|u\|^2_{L^2(\mathbb{R}_+)}}. \tag{2.5}$$

Let us denote by $\varphi_{\gamma,\xi}$ the positive (and L^2-normalized) first eigenfunction of $\mathcal{L}[\gamma,\xi]$. It is proved in [30] that the functions

$$(\gamma,\xi)\mapsto\lambda_1(\gamma,\xi),\quad (\gamma,\xi)\mapsto\varphi_{\gamma,\xi}\in L^2(\mathbb{R}_+)$$

are regular (i.e. of class C^∞), and we have the following formulas,

$$\partial_\xi\lambda_1(\gamma,\xi)=-\left(\lambda_1(\gamma,\xi)-\xi^2+\gamma^2\right)|\varphi_{\gamma,\xi}(0)|^2, \tag{2.6}$$

$$\partial_\gamma\lambda_1(\gamma,\xi)=|\varphi_{\gamma,\xi}(0)|^2. \tag{2.7}$$

Notice that (2.7) will yield that the function

$$(\gamma,\xi)\mapsto\varphi_{\gamma,\xi}(0)$$

is also regular of class C^∞.
We define the function :

$$\Theta(\gamma)=\inf_{\xi\in\mathbb{R}}\lambda_1(\gamma,\xi). \tag{2.8}$$

It is a result of [14] that there exists a unique $\xi(\gamma)>0$ such that,

$$\Theta(\gamma)=\lambda_1(\gamma,\xi(\gamma)), \tag{2.9}$$

and $\xi(\gamma)$ satisfies (cf. [30]),

$$\xi(\gamma)^2=\Theta(\gamma)+\gamma^2. \tag{2.10}$$

Moreover, the function $\Theta(\gamma)$ is of class C^∞ and satisfies,

$$\Theta'(\gamma)=|\varphi_\gamma(0)|^2, \tag{2.11}$$

where φ_γ is the positive (and L^2-normalized) eigenfunction associated to $\Theta(\gamma)$:

$$\varphi_\gamma=\varphi_{\gamma,\xi(\gamma)}. \tag{2.12}$$

When $\gamma=0$, we write,

$$\Theta_0:=\Theta(0),\quad \xi_0:=\xi(0). \tag{2.13}$$

We define also the universal constant C_1 by,

$$C_1:=\frac{|\varphi_0(0)|^2}{3}. \tag{2.14}$$

Another important fact is the following consequence of standard Sturm-Liouville theory.

Lemma 2.1. *For any $\xi\in\mathbb{R}$, we have,*

$$\lambda_2^N(\xi)>\lambda_1^D(\xi).$$

2.2 Boundary Coordinates

We recall now the definition of the standard coordinates that straightens a portion of the boundary $\partial\Omega$. Given $t_0>0$, let us introduce the following neighborhood of the boundary,

$$\mathcal{N}_{t_0}=\{x\in\mathbb{R}^2;\quad \mathrm{dist}(x,\partial\Omega)<t_0\}. \tag{2.15}$$

As the boundary is smooth, let $s \in]-\frac{|\partial\Omega|}{2}, \frac{|\partial\Omega|}{2}] \mapsto M(s) \in \partial\Omega$ be a regular parametrization of $\partial\Omega$ that satisfies :

$$\begin{cases} s \text{ is the oriented `arc length' between } M(0) \text{ and } M(s). \\ T(s) := M'(s) \text{ is a unit tangent vector to } \partial\Omega \text{ at the point } M(s). \\ \text{The orientation is positive, i.e. } \det(T(s), \nu(s)) = 1. \end{cases}$$

We recall that $\nu(s)$ is the unit outward normal of $\partial\Omega$ at the point $M(s)$. The scalar curvature κ_{r} is now defined by :

$$T'(s) = \kappa_{\mathrm{r}}(s)\nu(s). \tag{2.16}$$

When t_0 is sufficiently small, the map :

$$\Phi :]-|\partial\Omega|/2, |\partial\Omega|/2] \times]-t_0, t_0[\ni (s,t) \mapsto M(s) - t\nu(s) \in \mathcal{N}_{t_0}, \tag{2.17}$$

is a diffeomorphism. For $x \in \mathcal{N}_{t_0}$, we write,

$$\Phi^{-1}(x) := (s(x), t(x)), \tag{2.18}$$

where

$$t(x) = \operatorname{dist}(x, \partial\Omega) \text{ if } x \in \Omega \quad \text{and } t(x) = -\operatorname{dist}(x, \partial\Omega) \text{ if } x \notin \Omega.$$

The jacobian of the transformation Φ^{-1} is equal to,

$$a(s,t) = \det\left(D\Phi^{-1}\right) = 1 - t\kappa_{\mathrm{r}}(s). \tag{2.19}$$

To a vector field $A = (A_1, A_2) \in H^1(\mathbb{R}^2; \mathbb{R}^2)$, we associate the vector field

$$\tilde{A} = (\tilde{A}_1, \tilde{A}_2) \in H^1(]-|\partial\Omega|/2, |\partial\Omega|/2] \times]-t_0, t_0[; \mathbb{R}^2)$$

by the following relations :

$$\tilde{A}_1(s,t) = (1 - t\kappa_{\mathrm{r}}(s))\vec{A}(\Phi(s,t)) \cdot M'(s), \quad \tilde{A}_2(s,t) = \vec{A}(\Phi(s,t)) \cdot \nu(s). \tag{2.20}$$

We get then the following change of variable formulas.

Proposition 2.2. *Let $u \in H^1_A(\mathbb{R}^2)$ be supported in $\mathcal{N}_{t_0}$. Writing $\widetilde{u}(s,t) = u(\Phi(s,t))$, then we have :*

$$\int_\Omega |(\nabla - iA)u|^2\, dx = \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} \int_0^{t_0} \left[|(\partial_s - i\tilde{A}_1)\widetilde{u}|^2 + a^{-2}|(\partial_t - i\tilde{A}_2)\widetilde{u}|^2\right] a\, dsdt, \tag{2.21}$$

$$\int_{\Omega^c} |(\nabla - iA)u|^2\, dx = \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} \int_{-t_0}^{0} \left[|(\partial_s - i\tilde{A}_1)\widetilde{u}|^2 + a^{-2}|(\partial_t - i\tilde{A}_2)\widetilde{u}|^2\right] a\, dsdt, \tag{2.22}$$

and

$$\int_{\mathbb{R}^2} |u(x)|^2\, dx = \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} \int_{-t_0}^{t_0} |\widetilde{u}(s,t)|^2 a\, dsdt. \tag{2.23}$$

We have also the relation :

$$(\partial_{x_1} A_2 - \partial_{x_2} A_1)\, dx_1 \wedge dx_2 = \left(\partial_s \tilde{A}_2 - \partial_t \tilde{A}_1\right) a^{-1} ds \wedge dt,$$

which gives,

$$\operatorname{curl}_{(x_1,x_2)} A = (1 - t\kappa_{\mathrm{r}}(s))^{-1} \operatorname{curl}_{(s,t)} \tilde{A}.$$

We give in the next proposition a standard choice of gauge.

Proposition 2.3. *Consider a vector field $A = (A_1, A_2) \in C^1_{\rm loc}(\mathbb{R}^2; \mathbb{R}^2)$ such that*

$$\operatorname{curl} A = 1 \quad \text{in } \mathbb{R}^2.$$

For each point $x_0 \in \partial\Omega$, there exist a neighborhood $\mathcal{V}_{x_0} \subset \mathcal{N}_{t_0}$ of x_0 and a smooth real-valued function ϕ_{x_0} such that the vector field $A_{new} := A - \nabla\phi_{x_0}$ satisfies in $\mathcal{V}_{x_0}$:

$$\tilde{A}^2_{new} = 0, \tag{2.24}$$

and,

$$\tilde{A}^1_{new} = -t\left(1 - \frac{t}{2}\kappa_{\rm r}(s)\right), \tag{2.25}$$

with $\tilde{A}_{new} = (\tilde{A}^1_{new}, \tilde{A}^2_{new})$.

2.3 The Neumann and Dirichlet magnetic Schrödinger operators

Let us consider the differential operator,

$$\mathcal{P}_\varepsilon = -(\nabla - \varepsilon^{-2}\mathbf{F})^2,$$

where ε is a small parameter. Given a domain $U \subset \mathbb{R}^2$, we denote by $\mathcal{P}^N_{\varepsilon,U}$ and $\mathcal{P}^D_{\varepsilon,U}$ the Neumann and Dirichlet realizations of $\mathcal{P}_\varepsilon$ in U respectively. Then we introduce :

$$\mu^N(\varepsilon; U) = \inf \operatorname{Sp}\left(\mathcal{P}^N_{\varepsilon,U}\right), \quad \mu^D(\varepsilon; U) = \inf \operatorname{Sp}\left(\mathcal{P}^D_{\varepsilon,U}\right). \tag{2.26}$$

We recall the following result of [24].

Proposition 2.4. *Given a domain $U \subset \mathbb{R}^2$ with compact and smooth boundary, there exist constants $C, \varepsilon_0 > 0$ such that, if $\operatorname{curl}\mathbf{F} = 1$ in U, then we have for any $\varepsilon \in]0, \varepsilon_0]$,*

$$\varepsilon^{-2} \le \mu^D(\varepsilon; U) \le \varepsilon^{-2} + C\varepsilon^{-2}\exp\left(-\frac{1}{\varepsilon}\right), \tag{2.27}$$

$$\left|\mu^N(\varepsilon; U) - \Theta_0\varepsilon^{-2}\right| \le C\varepsilon^{-1}. \tag{2.28}$$

We define the quadratic form,

$$\mathcal{H}^1_{\varepsilon^{-2}\mathbf{F}}(U) \ni u \mapsto q_{\varepsilon,\mathbf{F},U}(u) = \|(\nabla - i\varepsilon^{-2}\mathbf{F})u\|^2_{L^2(U)}. \tag{2.29}$$

Proposition 2.5. *Suppose that U has a smooth compact boundary and that $\operatorname{curl}\mathbf{F} = 1$ in U.*

1. *If $u \in C_0^\infty(U)$, then*
$$q_{\varepsilon,\mathbf{F},U}(u) \ge \varepsilon^{-2}\|u\|^2_{L^2(U)}. \tag{2.30}$$

2. *Given a point $z_0 \in \partial U$, suppose that there exists a function $\phi_0 \in C^2(U)$ and a constant C_0 that depends only on U such that, upon putting,*
$$\mathbf{F}_{new} = \mathbf{F} + \nabla\phi_0,$$
we have in boundary coordinates,
$$\left|\widetilde{\mathbf{F}}_{new}(s,t) - A_0(s,t)\right| \le C_0 t^2, \quad \text{in a neighborhood of } z_0,$$

where $\widetilde{\mathbf{F}}_{new}$ is associated to $\mathbf{F}_{new}$ by the relation (2.20), and the magnetic potential A_0 is defined by

$$A_0(s,t) = (-t,0), \quad \forall (s,t) \in \mathbb{R}\times\mathbb{R}_+ .$$

Then there exist constants $C, \varepsilon_0 > 0$ depending only on U, such that, for all

$$\zeta_0, \rho, \theta > 0, \quad \varepsilon \in]0,\varepsilon_0], \quad u \in \mathcal{H}^1_{\varepsilon^{-2}\mathbf{F}}(U), \quad \operatorname{supp} u \subset D(z_0, \zeta_0\varepsilon^\rho),$$

we have,

$$\begin{aligned} q_{\varepsilon,\mathbf{F},U}(u) \quad \geq \quad & \left(1 - C\zeta_0\varepsilon^\rho - C\varepsilon^{2\theta}\right) q_{\varepsilon,A_0,\mathbb{R}\times\mathbb{R}_+}\left(\exp\left(-i\frac{\phi_0}{\varepsilon^2}\right)\widetilde{u}\right) \\ & -C\varepsilon^{4\rho-2\theta-4}\|\widetilde{u}\|^2_{L^2(\mathbb{R}\times\mathbb{R}_+)}, \end{aligned} \tag{2.31}$$

where the function $\widetilde{u}$ is defined by means of the change of variables (cf. (2.17)),

$$\widetilde{u}(s,t) = u\left(\Phi(s,t)\right).$$

The lower bound (2.30) is well-known (cf. [5]) and is easy to prove in our case. The estimate (2.31) is essentially obtained in [24, p. 16]. The proof consists actually of writing the quadratic form $q_{\varepsilon,\mathbf{F},U}$ in the coordinate system (s,t), then doing a gauge transformation that permits one to work with $\mathbf{F}_{new}$ which can be approximated by the 'canonical' magnetic potential A_0, and finally of applying a Cauchy-Schwarz inequality.

Finally we recall the definition of a useful 'scaled' partition of unity of $\mathbb{R}^2$ attached to a covering by balls of radius $\zeta_0\varepsilon^\rho$.

Proposition 2.6. *Let $0 \leq \rho \leq 2$ and $\varepsilon_0 > 0$. There exist a constant $C > 0$ and a partition of unity χ_j of $\mathbb{R}^2$ such that with,*

$$\chi_j^\varepsilon(x) := \chi_j\left(\frac{x}{\zeta_0\varepsilon^\rho}\right), \quad \varepsilon \in]0,\varepsilon_0] \quad \text{and}\, \zeta_0 > 0,$$

we have,

$$\sum_j |\chi_j^\varepsilon|^2 = 1, \tag{2.32}$$

$$\sum_j |\nabla\chi_j^\varepsilon|^2 \leq C\zeta_0^{-2}\varepsilon^{-2\rho}, \tag{2.33}$$

$$\operatorname{supp}\chi_j^\varepsilon \subset D(z_j^\varepsilon, \zeta_0\varepsilon^\rho) \text{ and } \left\{ \begin{array}{l} \text{either } \operatorname{supp}\chi_j^\varepsilon \cap \partial\Omega = \emptyset, \\ \text{or } z_j^\varepsilon \in \partial\Omega. \end{array}\right. \tag{2.34}$$

Moreover, we have the following decomposition formula,

$$\forall u \in \mathcal{H}^1_{\varepsilon^{-2}\mathbf{F}}(U), \quad q_{\varepsilon,\mathbf{F},U}(u) = \sum_j q_{\varepsilon,\mathbf{F},U}\left(\chi_j^\varepsilon u\right) - \sum_j \|\,|\nabla\chi_j^\varepsilon|u\|^2_{L^2(U)}, \tag{2.35}$$

where U is either Ω or $\overline{\Omega}^c$.

Formula (2.35) is called in other contexts the IMS formula, see however [13].

3 Analysis of the canonical 'interface' operator

The analysis of the half-plane model operator associated with the quadratic form (1.10) leads us to the analysis of a family of ordinary differential operators (This is by performing a partial Fourier transformation with respect to the second variable z).

3.1 Notations and preliminaries

Given $a, m, \alpha > 0$ and $\xi \in \mathbb{R}$, let us define the quadratic form :

$$B^1(\mathbb{R}) \ni u \mapsto Q[a,m,\alpha;\xi](u), \tag{3.1}$$

where :

$$\begin{aligned} Q[a,m,\alpha;\xi](u) = & \int_{\mathbb{R}_+} \left(|u'(t)|^2 + |(t-\xi)u(t)|^2 - \alpha|u(t)|^2\right) dt \\ & + \int_{\mathbb{R}_-} \left(\frac{1}{m}\left[|u'(t)|^2 + |(t-\xi)u(t)|^2\right] + a\alpha|u(t)|^2\right) dt. \end{aligned} \tag{3.2}$$

We denote by $H[a,m,\alpha;\xi]$ the self-adjoint operator associated to the quadratic form (3.1) by Friedrichs' Theorem. The domain of $H[a,m,\alpha;\xi]$ is defined by :

$$D(H[a,m,\alpha;\xi]) = \left\{ u \in B^1(\mathbb{R}); \quad u_{|\mathbb{R}_\pm} \in B^2(\mathbb{R}_\pm), \quad u'(0_+) = \frac{1}{m}u'(0_-) \right\}, \tag{3.3}$$

and for $u \in D(H[a,m,\alpha;\xi])$, we have,

$$(H[a,m,\alpha;\xi]u)(t) = \left\{ \begin{array}{ll} \left[\left(-\partial_t^2 + (t-\xi)^2 - \alpha\right) u\right](t); & \text{if } t > 0, \\ \\ \left[\left(\frac{1}{m}\left\{-\partial_t^2 + (t-\xi)^2\right\} + a\alpha\right) u\right](t); & \text{if } t < 0. \end{array} \right. \tag{3.4}$$

We denote by $\mu_1(a,m,\alpha;\xi)$ the first eigenvalue of $H[a,m,\alpha;\xi]$ which is given by the min-max principle,

$$\mu_1(a,m,\alpha;\xi) = \inf_{u \in B^1(\mathbb{R}), u \neq 0} \frac{Q[a,m,\alpha;\xi](u)}{\|u\|^2_{L^2(\mathbb{R})}}. \tag{3.5}$$

The eigenvalue $\mu_1(a,m,\alpha;\xi)$ is simple and there exists a unique strictly positive (and L^2-normalized) eigenfunction $f^{a,m}_{\alpha,\xi}$. To see that $f^{a,m}_{\alpha,\xi}(0) > 0$, we suppose for a contradiction that $f^{a,m}_{\alpha,\xi}(0) = 0$, then we define the function $v \in B^1(\mathbb{R})$ by :

$$v(t) = f^{a,m}_{\alpha,\xi}(t) \quad \text{if } t > 0, \quad v(t) = 0 \quad \text{if } t \leq 0.$$

So v is in the form domain of $Q[a,m,\alpha;\xi]$ and an integration by parts yields the equality :

$$\frac{Q[a,m,\alpha;\xi](v)}{\|v\|^2_{L^2(\mathbb{R})}} = \mu_1(a,m,\alpha;\xi).$$

Therefore, by the min-max principle, v is an eigenfunction of $H[a,m,\alpha;\xi]$ and consequently $v \in D(H[a,m,\alpha;\xi])$. Hence $v'(0_+) = v(0_+) = 0$ and so we get by Cauchy's Uniqueness Theorem for solutions of ordinary differential equations that $v \equiv 0$, which is the desired contradiction.
We can now define the following 'effective de Gennes parameter' :

$$\gamma(a,m,\alpha;\xi) := \left(\frac{\left(f^{a,m}_{\alpha,\xi}\right)'}{f^{a,m}_{\alpha,\xi}} \right)(0_+). \tag{3.6}$$

Using the boundary condition satisfied by $f^{a,m}_{\alpha,\xi}$, we get,

$$\gamma(a,m,\alpha;\xi) = \frac{1}{m} \left(\frac{\left(f^{a,m}_{\alpha,\xi}\right)'}{f^{a,\alpha}_{\alpha,\xi}} \right)(0_-).$$

In the physical literature [19, 26, 15], the parameter $\frac{1}{\gamma(a,m,\alpha;\xi)}$ is usually called the 'extrapolation length'.

Let us notice that the quadratic form $Q[a,m,\alpha;\xi]$ has a fixed domain and that, given u in its form domain, the function

$$\mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R} \ni (a,m,\alpha;\xi) \mapsto Q[a,m,\alpha;\xi](u)$$

is analytic. Therefore, $H[a,m,\alpha;\xi]$ is a 'holomorphic operator of Type A' with respect to the variables a, m, α, ξ (cf. [32])[5]. We then get, thanks to the simplicity of $\mu_1(a,m,\alpha;\xi)$, that the first eigenvalue and the corresponding normalized eigenfunction of $Q[a,m,\alpha;\xi]$ are analytic. This yields the following regularity result, proved in [29, Theorem 3.1.2] by a different approach.

Theorem 3.1. *The functions,*

$$(a,m,\alpha,\xi) \mapsto \mu_1(a,m,\alpha;\xi), \quad (a,m,\alpha,\xi) \mapsto f_{\alpha,\xi}^{a,m} \in L^2(\mathbb{R}),$$

and

$$(a,m,\alpha,\xi) \mapsto \gamma(a,m,\alpha;\xi)$$

are of class C^∞.

Remark 3.2. *Notice that, by a partial Fourier transformation (with respect to the variable z) and by separation of variables (cf. [36]), one has, (cf. (1.13)),*

$$\mu^{(1)}(\kappa, H; \mathbb{R}\times\mathbb{R}_+) = \kappa H \left(\inf_{\xi\in\mathbb{R}} \mu_1 \left(a, m, \frac{\kappa}{H}; \xi \right) \right). \tag{3.7}$$

3.2 Variation with respect to ξ

We study in this section the variations of the function :

$$\xi \mapsto \mu_1(a,m,\alpha;\xi).$$

It results from the min-max principle the following lemma.

Lemma 3.3. *For all $a,m,\alpha > 0$ and $\xi \in \mathbb{R}$, the eigenvalue $\mu_1(a,m,\alpha;\xi)$ satisfies :*

$$\begin{aligned} &\min\left(\lambda_1^N(\xi) - \alpha, \frac{1}{m}\lambda_1^N(-\xi) + a\alpha\right) \\ &\leq \mu_1(a,m,\alpha;\xi) \\ &\leq \min\left(\lambda_1^D(\xi) - \alpha, \frac{1}{m}\lambda_1^D(-\xi) + a\alpha\right). \end{aligned} \tag{3.8}$$

Combining with previous well known results about $\lambda_1^N(\cdot)$ and $\lambda_1^D(\cdot)$ (cf. [8, 24]), we get that the function $\mu_1(a,m,\alpha;\cdot)$ is bounded and,

$$\lim_{\xi\to-\infty} \mu_1(a,m,\alpha;\xi) = \frac{1}{m} + a\alpha, \quad \lim_{\xi\to+\infty} \mu_1(a,m,\alpha;\xi) = 1-\alpha. \tag{3.9}$$

In the next proposition, we give an explicit formula for the derivative of $\mu_1(a,m,\alpha;\cdot)$.

[5] In [32], the analytic perturbation theory is developed for operators depending on a single variable. However it is straightforward to generalize this theory to operators depending on 'many variables' (see for example [8]) and the analyticity of the eigenvalues will remain true when they are simple.

Proposition 3.4. *For all $a, m, \alpha > 0$ and $\xi \in \mathbb{R}$, we have :*

$$\begin{aligned}&\partial_\xi \mu_1(a,m,\alpha;\xi)\\ &= \left[(m-1)|\gamma(a,m,\alpha;\xi)|^2 + \left(1-\tfrac{1}{m}\right)\xi^2 - (1+a)\alpha\right]\left|f_{\alpha,\xi}^{a,m}(0)\right|^2,\end{aligned} \tag{3.10}$$

where $\gamma(a,m,\alpha;\xi)$ is defined in (3.6).

Proof. We follow the method of Dauge-Helffer [14]. To simplify, we shall omit the reference to the parameters a, m, α and write $\mu_1(-;\xi)$, f_ξ and $H[-;\xi]$ respectively for the first eigenvalue $\mu_1(a,m,\alpha;\xi)$, the first eigenfunction $f_{\alpha,\xi}^{a,m}$ and the operator $H[a,m,\alpha;\xi]$.
Consider a real $\tau > 0$. Notice that :

$$H[-;\xi]f_{\xi+\tau}(t+\tau) = \mu_1(-;\xi+\tau)f_{\xi+\tau}(t+\tau), \quad (t \notin [-\tau, 0]).$$

An integration by parts yields :

$$\begin{aligned}&\int_{-\infty}^{-\tau} H[-;\xi]f_\xi(t)\, f_{\xi+\tau}(t+\tau)\, dt = \int_{-\infty}^{-\tau} f_\xi(t) H[-;\xi] f_{\xi+\tau}(t+\tau)\, dt\\ &+\frac{1}{m}\left\{f_\xi(-\tau)f'_{\xi+\tau}(0_-) - f'_\xi(-\tau)f_{\xi+\tau}(0)\right\},\end{aligned} \tag{3.11}$$

and,

$$\begin{aligned}&\int_0^{+\infty} H[-;\xi]f_\xi(t)\, f_{\xi+\tau}(t+\tau)\, dt = \int_0^{+\infty} f_\xi(t) H[-;\xi] f_{\xi+\tau}(t+\tau)\, dt\\ &-f_\xi(0)f'_{\xi+\tau}(\tau) + f'_\xi(0_+)f_{\xi+\tau}(\tau).\end{aligned} \tag{3.12}$$

By taking the sum of the preceding two equalities, we get the following identity :

$$\begin{aligned}&(\mu_1(-;\xi+\tau) - \mu_1(-;\xi)) \int_{-\infty}^{+\infty} f_\xi(t) f_{\xi+t}(t+\tau) dt\\ &= \frac{1}{m}\left\{f'_\xi(-\tau)f_{\xi+\tau}(0) - f_\xi(-\tau)f'_{\xi+\tau}(0_-)\right\}\\ &\quad + f_\xi(0)f'_{\xi+\tau}(\tau) - f'_\xi(0+)f_{\xi+\tau}(\tau)\\ &\quad + (\mu_1(-;\xi+\tau) - \mu_1(-;\xi)) \int_{-\tau}^{0} f_\xi(t) f_{\xi+t}(t+\tau) dt.\end{aligned} \tag{3.13}$$

Using the mean value theorem, we get,

$$\begin{aligned}&\left|f'_\xi(-\tau) - f'_\xi(0_-) + \tau f''_\xi(0_-)\right| \leq \tau^2 \|f'''_\xi\|_{L^\infty(0,1)},\\ &\left|f_{\xi+\tau}(\tau) - f_{\xi+\tau}(0) - \tau f'_{\xi+\tau}(0_+)\right| \leq \tau^2 \|f''_{\xi+\tau}\|_{L^\infty(0,1)}.\end{aligned}$$

By the boundary condition $f'_\xi(0_+) = \frac{1}{m} f'_\xi(0_-)$, we can rewrite the preceding estimate in the form :

$$\begin{aligned}&\frac{1}{m} f'_\xi(-\tau) f_{\xi+\tau}(0) - f'_\xi(0_+) f_{\xi+\tau}(\tau)\\ &= -\tau f'_\xi(0_+) f'_{\xi+\tau}(0_+) - \frac{\tau}{m} f_{\xi+\tau}(0) f''_\xi(0_-) + \mathcal{O}(\tau^2).\end{aligned}$$

The same argument yields :

$$\begin{aligned}&f_\xi(0) f'_{\xi+\tau}(\tau) - \frac{1}{m} f_\xi(-\tau) f'_{\xi+\tau}(0_-)\\ &= m\tau f'_\xi(0_+) f'_{\xi+\tau}(0_+) + \tau f_\xi(0) f''_{\xi+\tau}(0_+) + \mathcal{O}(\tau^2).\end{aligned}$$

Dividing the two sides of (3.12) by τ then taking the limit $\tau \to 0$, we get :

$$\partial_\xi \mu_1(-;\xi) = (m-1)\,|f'_\xi(0_+)|^2 - \frac{1}{m} f_\xi(0) f''_\xi(0_-) + f_\xi(0) f''_\xi(0_+). \tag{3.14}$$

The boundary condition and the equations satisfied by $f_\xi(t)$ in $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively permit now to deduce Formula (3.10). □

Let us define the set of points of minima of $\mu_1(a,m,\alpha;\xi)$,

$$\mathrm{M}(a,m,\alpha) = \{\eta \in \mathbb{R}; \quad \mu_1(a,m,\alpha;\eta) = \inf_{\xi\in\mathbb{R}} \mu_1(a,m,\alpha;\xi)\}. \tag{3.15}$$

Formula (3.10) permits to obtain sufficient conditions on a, m, α for the set $\mathrm{M}(a,m,\alpha)$ to be empty or not.

Theorem 3.5. *If $m \leq 1$, then the set $\mathrm{M}(a,m,\alpha)$ is empty and*

$$\inf_{\xi\in\mathbb{R}} \mu_1(a,m,\alpha;\xi) = 1-\alpha. \tag{3.16}$$

On the other hand, let $\epsilon_0 \in]0, a\alpha[$ and suppose that a, m, α satisfy,

$$-\epsilon_0 \leq \inf_{\xi\in\mathbb{R}} \mu_1(a,m,\alpha;\xi) \leq \epsilon_0, \tag{3.17}$$

then, if $m > 1$, the set $\mathrm{M}(a,m,\alpha)$ is non-empty. Moreover, if $\xi \in \mathrm{M}(a,m,\alpha)$, then,

$$|\xi| \leq \sqrt{(1+a)\left(1-\frac{1}{m}\right)^{-1} \alpha}, \tag{3.18}$$

and the eigenfunction $f^{a,m}_{\alpha,\xi}$ satisfies :

$$\left(f^{a,m}_{\alpha,\xi}\right)'(0_\pm) > 0.$$

Proof. Notice that, if $m \leq 1$, Formula (3.10) implies that the function $\mapsto \mu_1(a,m,\alpha;\cdot)$ is strictly decreasing. Therefore,

$$\mu_1(a,m,\alpha;\xi) = \lim_{\xi\to+\infty} \mu^{(1)}(a,m,\alpha;\xi),$$

which proves (3.16), thanks to (3.9).
Suppose now that $m > 1$ and that Hypothesis (3.17) holds. We denote by :

$$d = \sqrt{(1+a)\left(1-\frac{1}{m}\right)^{-1} \alpha}. \tag{3.19}$$

Notice that there exists $\xi_d \in [-d, d]$ such that,

$$\mu_1(a,m,\alpha;\xi_d) = \min_{\xi\in[-d,d]} \mu_1(a,m,\alpha;\xi).$$

As $m > 1$, Formula (3.10) gives that the function $\mu_1(a,m,\alpha;\cdot)$ is strictly increasing on the intervals $]-\infty, -d]$ and $[d, +\infty[$. We then have,

$$\mu_1(a,m,\alpha;\xi_d) \leq \mu_1(a,m,\alpha,d) < \mu_1(a,m,\alpha;\xi), \quad \forall \xi \in [d, +\infty],$$

and

$$\mu_1(a,m,\alpha;\xi) > \frac{1}{m} + a\alpha > \epsilon_0, \quad \forall \xi \in]-\infty,-d].$$

This gives now,

$$\mu_1(a,m,\alpha;\xi_d) = \inf_{\xi\in\mathbb{R}} \mu_1(a,m,\alpha;\xi),$$

and hence the set $\mathrm{M}(a,m,\alpha)$ is non-empty and bounded.
Given $\xi \in \mathrm{M}(a,m,\alpha)$, the equation satisfied by $f^{a,m}_{\alpha,\xi}$ in $\mathbb{R}_-$ can be written as :

$$\begin{cases} -\left(f^{a,m}_{\alpha,\xi}\right)'' + (t-\xi)^2 f^{a,m}_{\alpha,\xi} = m\left(\mu_1(a,m,\alpha;\xi) - a\alpha\right) f^{a,m}_{\alpha,\xi}, \quad t<0, \\ \left(f^{a,m}_{\alpha,\xi}\right)'(0_-) = \gamma(a,m,\alpha;\xi) f^{a,m}_{\alpha,\xi}(0), \end{cases} \tag{3.20}$$

with $\mu_1(a,m,\alpha;\xi) - a\alpha < \epsilon_0 - a\alpha < 0$. Therefore $\gamma(a,m,\alpha;\xi) > 0$ and consequently $\left(f^{a,m}_{\alpha,\xi}\right)'(0_\pm) > 0$. □

We collect some useful relations in the next proposition.

Proposition 3.6. *Let $\eta \in \mathrm{M}(a,m,\alpha)$. Then we have,*

$$\int_{\mathbb{R}_+} (t-\eta)\left|f^{a,m}_{\alpha,\eta}(t)\right|^2 dt + \frac{1}{m}\int_{\mathbb{R}_-} (t-\eta)\left|f^{a,m}_{\alpha,\eta}(t)\right|^2 dt = 0, \tag{3.21}$$

$$\begin{aligned} &\int_{\mathbb{R}_+} (t-\eta)^3\left|f^{a,m}_{\alpha,\eta}(t)\right|^2 dt + \frac{1}{m}\int_{\mathbb{R}_-} (t-\eta)^3\left|f^{a,m}_{\alpha,\eta}(t)\right|^2 dt \\ &= \frac{1}{6}\left(1-\frac{1}{m}\right)\left|f^{a,m}_{\alpha,\eta}(0)\right|^2 + 2\eta^2\left(\eta^2\left(1-\frac{1}{m}\right) - (a+1)\alpha\right)\left|f^{a,m}_{\alpha,\eta}(0)\right|^2 \\ &+\frac{1}{3}\left(a-\frac{1}{m}\right)\alpha\int_{\mathbb{R}_-}\left|f^{a,m}_{\alpha,\eta}(t)\right|^2 dt. \end{aligned} \tag{3.22}$$

Proof. We denote by :

$$L_+ = -\partial_t^2 + (t-\eta)^2 - \alpha, \quad L_- = \frac{1}{m}\left(-\partial_t^2 + (t-\eta)^2\right) + a\alpha.$$

Let p a polynomial, $f = f^{a,m}_{\alpha,\eta}$ and $v = 2pf' - p'f$. By direct calculation, we have :

$$\begin{aligned} L_+ v &= \Big(p^{(3)} - 4p'\left[(t-\eta)^2 - \alpha\right] - 4p(t-\eta)\Big) f, \\ L_- v &= \frac{1}{m}\Big(p^{(3)} - 4p'\left[(t-\eta)^2 + ma\alpha\right] - 4(t-\eta)\Big) f. \end{aligned}$$

Integrating by parts, we get,

$$\begin{aligned} &\int_{\mathbb{R}_+} f(t)\cdot(L_+f)(t)\,dt + \frac{1}{m}\int_{\mathbb{R}_-} f(t)\cdot(L_-f)(t)\,dt \\ &= \left(v'(0_+) - v'(0_-)\right) f(0) - \left(v(0_+) - v(0_-)\right) f'(0_+). \end{aligned} \tag{3.23}$$

Taking $p=1$, we obtain (3.21). To obtain (3.22), we take $p = (t-\eta)^2$. In this case,

$$\begin{aligned} &\int_{\mathbb{R}_+} f(t)\cdot(L_+f)(t)\,dt + \frac{1}{m}\int_{\mathbb{R}_-} f(t)\cdot(L_-f)(t)\,dt \\ &= -2\left(1-\frac{1}{m}\right)|f(0)|^2 + 2\eta^2\left(f''(0_+) - \frac{1}{m}f''(0_-)\right) f(0), \end{aligned}$$

which is enough to deduce (3.22). □

3.3 The function $\alpha(a,m)$

In the theorem below, we define the function $\alpha(\cdot,\cdot)$ appearing in the statement of Theorem 1.1.

Theorem 3.7. *Given $a,m>0$, there exists a unique solution $\alpha(a,m)$ of the equation :*

$$\inf_{\xi\in\mathbb{R}} \mu_1\left(a,m,\alpha(a,m);\xi\right)=0. \tag{3.24}$$

Moreover, $\alpha(a,m)=1$ if $m\leq 1$, and $\Theta_0<\alpha(a,m)<1$ if $m>1$.

Proof. We start by proving the existence of $\alpha(a,m)$. The min-max principle gives that the function

$$\alpha\mapsto \inf_{\xi\in\mathbb{R}}\mu_1(a,m,\alpha;\xi)$$

is Lipschitz. Lemma 3.3 gives immediately that

$$\inf_{\xi\in\mathbb{R}}\mu_1(a,m,\alpha;\xi)>0,\quad \forall\alpha<\Theta_0,$$
$$\inf_{\xi\in\mathbb{R}}\mu_1(a,m,\alpha;\xi)<0,\quad \forall\alpha>1.$$

Therefore, by the intermediate value theorem, there exists at least one solution $\alpha=\alpha(a,m)\in[\Theta_0,1]$ satisfying (3.24).
If $m\leq 1$, then by (3.16), $\alpha(a,m)=1$ and hence it is unique. If $m>1$, the function $\mu_1(a,m,\alpha;\cdot)$ is increasing in $[d,+\infty[$ (cf. (3.10) and (3.19)) and thus, thanks to (3.9), we have :

$$\mu_1(a,m,\alpha;\xi)<1-\alpha,\quad \forall\xi\in[d,+\infty[,\quad \forall\alpha>0.$$

Consequently any solution of (3.24) satisfies

$$\alpha(a,m)<1.$$

That $\alpha(a,m)>\Theta_0$ follows from the fact that $\mathrm{M}(a,m,\alpha)$ is non-empty (cf. Theorem 3.5). Actually, let $\xi\in\mathrm{M}(a,m,\alpha)$ and let us look at the equation satisfied by the eigenfunction $f_{\alpha,\xi}^{a,m}$ in $\mathbb{R}_+$:

$$\left\{\begin{array}{l} -\partial_t^2 f_{\alpha,\xi}^{a,m}+(t-\xi)^2 f_{\alpha,\xi}^{a,m}=\alpha(a,m)f_{\alpha,\xi}^{a,m},\quad t>0,\\ \left(f_{\alpha,\xi}^{a,m}\right)'(0_+)=\gamma(a,m,\alpha;\xi)f_{\alpha,\xi}^{a,m}(0).\end{array}\right. \tag{3.25}$$

Then,

$$\alpha(a,m)\geq\lambda_1(\gamma(a,m,\alpha;\xi),\xi),$$

where $\lambda_1(\gamma(a,m,\alpha;\xi),\xi)$ is defined by (2.5). Theorem 3.5 gives $\gamma(a,m,\alpha;\xi)>0$, and therefore $\alpha(a,m)>\Theta_0$.
It remains now to prove the uniqueness of $\alpha(a,m)$. Actually, we need only to prove uniqueness when $m>1$. It is sufficient to prove the following claim,

$$\text{If }\inf_{\xi\in\mathbb{R}}\mu_1(a,m,\alpha;\xi)=0,\ \text{then }\forall\beta>\alpha,\quad \inf_{\xi\in\mathbb{R}}\mu_1(a,m,\beta;\xi)<0. \tag{3.26}$$

Notice that, after comparing the quadratic forms $Q[a,m,\alpha;\xi]$ and $Q[a,m,\beta;\xi]$, we get for any $\eta\in\mathrm{M}(a,m,\alpha)$,

$$Q[a,m,\beta;\eta](f_{\alpha,\eta}^{a,m})+(\beta-\alpha)\left(\int_{\mathbb{R}_+}|f_{\alpha,\eta}^{a,m}(t)|^2dt-a\int_{\mathbb{R}_-}|f_{\alpha,\eta}^{a,m}(t)|^2dt\right)=0. \tag{3.27}$$

We have the following immediate consequence of the min-max principle,

$$
\begin{aligned}
&\left(\lambda_1^N(\eta)-\alpha\right)\int_{\mathbb{R}_+}|f^{a,m}_{\alpha,\eta}(t)|^2dt+\left(\frac{1}{m}\lambda_1^N(-\eta)+a\alpha\right)\int_{\mathbb{R}_-}|f^{a,m}_{\alpha,\eta}(t)|^2dt\\
&\leq \mu_1(a,m,\alpha;\eta)\int_{\mathbb{R}}|f^{a,m}_{\alpha,\eta}(t)|^2dt.
\end{aligned}
$$

Since $\mu_1(a,m,\alpha;\eta)=0$ and $m>1$, the above estimate reads as

$$
\frac{\Theta_0}{m}\leq\left(\int_{\mathbb{R}_+}|f^{a,m}_{\alpha,\eta}(t)|^2dt-a\int_{\mathbb{R}_-}|f^{a,m}_{\alpha,\eta}(t)|^2dt\right).
$$

We have actually used the fact that $\Theta_0<\alpha<1$ and that $\lambda_1^N(\pm\eta)\geq\Theta_0$.
Therefore, thanks to (3.27), we get finally,

$$
\mu_1(a,m,\beta;\eta)<0,
$$

which proves Claim (3.26). □

In the next proposition, we establish the monotonicity of $\alpha(a,\cdot)$.

Proposition 3.8. *The function* $[1,+\infty[\ni m\mapsto\alpha(a,m)$ *is strictly decreasing.*

Proof. As we are interested in the dependence on m, we omit a from the notation and we write $\alpha(m)$ for $\alpha(a,m)$. Suppose that :

$$
m_1>m_2>1.
$$

Notice that, $\forall\xi\in\mathbb{R}$, $\forall u\in B^1(\mathbb{R})$, we have :

$$
\begin{aligned}
Q[a,m_1,\alpha(m_1);\xi](u) &= Q[a,m_2,\alpha(m_2);\xi](u)\\
&\quad+\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\int_{\mathbb{R}_+}\left\{|u'(t)|^2+|(t-\xi)u(t)|^2\right\}dt\\
&\quad+(\alpha(m_2)-\alpha(m_1))\left(\int_{\mathbb{R}_+}|u(t)|^2dt-a\int_{\mathbb{R}_-}|u(t)|^2dt\right).
\end{aligned}
$$

In particular, for $\xi=\eta\in\mathrm{M}(a,m_2,\alpha)$ and $u=f_{\alpha(m_2)}$, we have :

$$
\begin{aligned}
&Q[a,m,\alpha(m_1);\eta](f_{\alpha(m_2)})\leq\Theta_0\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\int_{\mathbb{R}_+}|f_{\alpha(m_2)}(t)|^2dt\\
&+(\alpha(m_2)-\alpha(m_1))\left(\int_{\mathbb{R}_+}|f_{\alpha(m_2)}(t)|^2dt-a\int_{\mathbb{R}_-}|f_{\alpha(m_2)}(t)|^2dt\right),
\end{aligned}
$$

where $f_{\alpha(m_2)}=f^{a,m_2}_{\alpha(m_2),\eta}$.
Suppose by contradiction that $\alpha(m_1)\geq\alpha(m_2)$, we obtain, thanks to (3.27) and the fact that $\alpha(m_2)<1$,

$$
\begin{aligned}
Q[a,m,\alpha(m_1);\eta](f_{\alpha(m_2)}) &\leq \Theta_0\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\int_{\mathbb{R}_+}|f_{\alpha(m_2)}(t)|^2dt\\
&\quad+(\alpha(m_2)-\alpha(m_1))\frac{\Theta_0}{m}.
\end{aligned}
$$

The min-max principle now gives :

$$
\inf_{\xi\in\mathbb{R}}\mu_1(a,m_1,\alpha(m_1);\xi)<0,
$$

which is the desired contradiction. □

3.4 Asymptotic analysis with respect to m

We finish in this subsection the proof of Theorem 1.2. We start with the following technical lemma.

Lemma 3.9. *Given $a,m>0$, let $\alpha=\alpha(a,m)$. For any $\epsilon\in]0,1[$, there exists a constant $C>0$, such that,*

$$\forall m>1,\quad \forall a>0,\quad \forall \xi\in \mathrm{M}(a,m,\alpha),$$

the eigenfunction $f_{\alpha,\xi}^{a,m}$ decays in the following way,

$$\left\|\exp\left(\frac{\epsilon(t-\xi)^2}{2}\right)f_{\alpha,\xi}^{a,m}\right\|_{H^1(\mathbb{R}_+)}+\frac{1}{\sqrt{m}}\left\|\exp\left(\frac{\epsilon(t-\xi)^2}{2}\right)f_{\alpha,\xi}^{a,m}\right\|_{H^1(\mathbb{R}_-)}\leq C. \tag{3.28}$$

Proof. Let us consider a function $\Phi\in H^1(\mathbb{R})$. For simplicity we shall write f for $f_{\alpha,\xi}^{a,m}$. Given an integer $N\in\mathbb{N}$, an integration by parts gives the following identity :

$$\begin{aligned}&\int_0^N\left[\left|(e^\Phi f)'\right|^2+\left|(t-\xi)e^\Phi f\right|^2-\alpha\left|e^\Phi f\right|^2\right]dt-f'(N)e^{2\Phi(N)}f(N)\\&\int_{-N}^0\left[\frac{1}{m}\left(\left|(e^\Phi f)'\right|^2+\left|(t-\xi)e^\Phi f\right|^2\right)+a\alpha\left|e^\Phi f\right|^2\right]dt+f'(-N)e^{\Phi(-N)}f(-N)\\&=\mu_1(a,m,\alpha;\xi)\left\|e^\Phi f\right\|^2_{L^2([-N,N])}+\left\|\Phi'e^\Phi f\right\|^2_{L^2([0,N])}+\frac{1}{m}\left\|\Phi'e^\Phi f\right\|^2_{L^2([-N,0])}.\end{aligned} \tag{3.29}$$

Let us recall that the eigenfunction f is strictly positive and that α and ξ are selected in such a way that $\mu_1(a,m,\alpha;\xi)=0$. It results then from the eigenvalue equation satisfied by f :

$$f''(t)>0,\quad \forall t\in]-\infty,\xi[\cup]\xi+\sqrt{\alpha},+\infty[.$$

Therefore, the function f' is increasing on $]-\infty,\xi[\cup]\xi+\sqrt{\alpha},+\infty[$. On the other hand, as $f_{|\mathbb{R}_\pm}\in H^2(\mathbb{R}_\pm)$, the Sobolev Imbedding Theorem gives $\lim_{t\to+\infty}f'(t)=0$. Thus, combining with the monotonicity of f', we get,

$$f'(\pm N)<0,\quad \forall N>\max(\xi+\sqrt{\alpha},-\xi).$$

It results now from (3.29) the following estimate :

$$\begin{aligned}&\int_0^N\left[\left|(e^\Phi f)'\right|^2+\left((t-\xi)^2-\alpha-|\Phi'|\right)\left|e^\Phi f\right|^2\right]dt\\&+\int_{-N}^0\left[\left(\left|(e^\Phi f)'\right|^2+\left((t-\xi)^2-|\Phi'|^2\right)e^\Phi f\right)+a\alpha\left|e^\Phi f\right|^2\right]dt\leq 0.\end{aligned} \tag{3.30}$$

Now we take Φ as :

$$\Phi(t)=\epsilon\frac{(t-\xi)^2}{2}.$$

We can then rewrite (3.30) as :

$$\begin{aligned}&\int_{t\in[0,N],(t-\xi)\geq a_\epsilon}\left[\left|(e^\Phi f)'\right|^2+\left|e^\Phi f\right|^2\right]dt\\&+\frac{1}{m}\int_{t\in[-N,0],(t-\xi)\geq a_\varepsilon}\left[\left|(e^\Phi f)'\right|^2+\left|e^\Phi f\right|^2\right]dt\leq e^{\epsilon a_\epsilon/2},\end{aligned} \tag{3.31}$$

where $a_\epsilon > 0$ satisfies :

$$(1-\varepsilon^2)a_\epsilon^2 - 1 \geq 1.$$

Let us now take $C = e^{\epsilon a_\epsilon/2}$. Noticing that the estimate (3.31) is uniform with respect to N, we get the result of the lemma by passing to the limit $N \to +\infty$. □

In the next proposition we give a 'rough' asymptotic result.

Proposition 3.10. *Given $a > 0$, the following asymptotics hold :*

$$\lim_{m\to 1^+} \alpha(a,m) = 1 \quad \textit{and} \quad \lim_{m\to+\infty} \alpha(a,m) = \Theta_0.$$

Moreover, there exists a function $\epsilon(m)$ such that,

$$\epsilon(m) > 0, \quad \lim_{m\to+\infty} \epsilon(m) = 0 \text{ and } \mathrm{M}\,(a,m,\alpha(a,m)) \subset]-\epsilon(m)+\xi_0, \xi_0+\epsilon(m)[. \tag{3.32}$$

Proof. In the proof of this proposition and in the sequel of this section, we shall write α for $\alpha(a,m)$.

The limit $m \to 1^+$.

Let $\epsilon \in]0,1[$ and $m = 1+\epsilon$. Noticing that $\frac{1}{m} > 1-\epsilon$, we get for any function $u \in B^1(\mathbb{R})$,

$$\begin{aligned} Q[a,m,\alpha;\xi](u) \;\geq\; & \int_{\mathbb{R}} \left(|u'(t)|^2 + |(t-\xi)u(t)|^2 - \alpha|u(t)|^2\right) dt \qquad (3.33)\\ & -\epsilon\int_{\mathbb{R}_-} \left(|u'(t)|^2 + |(t-\xi)u(t)|^2\right) dt. \end{aligned}$$

(3.34)

Using the fact (cf. (2.13))

$$\int_{\mathbb{R}_+} \left(|u'(t)|^2 + |(t-\xi)u(t)|^2\right) dt \geq \Theta_0 \int_{\mathbb{R}_+} |u(t)|^2 dt,$$

together with $\alpha > \Theta_0$, we rewrite (3.33) as,

$$\begin{aligned} (1+m\epsilon)Q[a,m,\alpha;\xi](u) \;\geq\; & \int_{\mathbb{R}} \left[|u'(t)|^2 + |(t-\xi)u|^2\right] dt - \alpha \int_{\mathbb{R}} |u(t)|^2 dt \\ & + m\epsilon(\Theta_0 - \alpha)\int_{\mathbb{R}} |u(t)|^2 dt. \end{aligned}$$

Applying the min-max principle, we obtain,

$$(1+m\epsilon)\mu_1(a,m,\alpha;\xi) \geq 1-\alpha + m\epsilon(\Theta_0-\alpha).$$

Taking the infimum with respect to ξ, we obtain, thanks to the definition of α in Theorem 3.7,

$$0 \geq 1-\alpha+m(\Theta_0-\alpha)\epsilon.$$

Recalling that $\alpha < 1$, this is sufficient to deduce the required limit.

The limit $m \to +\infty$.

Recall that φ_0 is the first eigenfunction of $\mathcal{L}^N[\xi_0]$ (cf. Subsection 2.1). Let $\widetilde{\varphi}_0$ be the even extension of φ_0 in $\mathbb{R}$, i.e.

$$\widetilde{\varphi}_0(t) = \begin{cases} \varphi_0(t), & t > 0 \\ \varphi_0(-t), & t < 0. \end{cases}$$

Let χ be a cut-off function such that

$$0 \leq \chi \leq 1, \quad \chi = 0 \quad \text{in }]-\infty, -1], \quad \text{and } \chi = 1 \quad \text{in } [0, +\infty[. \tag{3.35}$$

There exist constants $C, m_0 > 0$ such that,

$$Q[a, m, \alpha; \xi_0] \left(\chi(\sqrt{m}\, t)\, \widetilde{\varphi}_0\right) \leq \Theta_0 - \alpha + \frac{C}{\sqrt{m}}, \quad \forall m \geq m_0. \tag{3.36}$$

It results now from the min-max principle and our choice of α that

$$Q[a, m, \alpha; \xi_0] \left(\chi(\sqrt{m}\, t)\, \widetilde{\varphi}_0\right) \geq 0.$$

Therefore, (3.36) reads,

$$0 \leq \Theta_0 - \alpha + \frac{C}{\sqrt{m}}, \quad \forall m \geq m_0. \tag{3.37}$$

Remembering that $\alpha > \Theta_0$, we get finally that,

$$\Theta_0 < \alpha \leq \Theta_0 + \frac{C}{\sqrt{m}},$$

and consequently, $\lim\limits_{m \to +\infty} \alpha(a, m) = \Theta_0$.

Localization of $\mathrm{M}(a, m, \alpha)$.

Let $\xi \in \mathrm{M}(a, m, \alpha(a, m))$. We denote by $\gamma_\xi = \gamma(a, m, \alpha(a, m); \xi)$. The equation $\partial_\xi \mu_1(a, m, \alpha; \xi) = 0$ gives the following relation :

$$\gamma_\xi^2 = \frac{a+1}{m-1}\alpha - \frac{1}{m}\xi^2. \tag{3.38}$$

Therefore, thanks to Theorem 3.5, there exist constants C, m_0 such that,

$$0 < \gamma_\xi \leq \frac{C}{\sqrt{m}}, \quad \forall m \geq m_0. \tag{3.39}$$

We define the function :

$$\phi_\xi = \frac{1}{\|f_{\alpha,\xi}^{a,m}\|_{L^2(\mathbb{R}_+)}} e^{-\gamma_\xi t} f_{\alpha,\xi}^{a,m}(t).$$

Using Lemma 3.9, we get,

$$\forall \delta \in]0, 1/2[, \quad \exists\, C_\delta, m_\delta > 0 \text{ s.t. } \forall m \geq m_\delta, \quad 1 - C_\delta m^{\delta - 1/2} \leq \|\phi_\xi\|_{L^2(\mathbb{R}_+)} \leq 1. \tag{3.40}$$

It results also from the eigenvalue equation satisfied by $f_{\alpha,\xi}^{a,m}$,

$$\left\{ \begin{array}{l} -\phi_\xi''(t) + (t-\xi)^2 \phi_\xi(t) = \alpha(a, m)\phi_\xi(t) + \gamma_\xi^2 \phi_\xi(t) + 2\gamma_\xi \phi_\xi'(t), \quad t > 0, \\ \phi_\xi'(0) = 0. \end{array} \right. \tag{3.41}$$

This yields the following estimate (cf. (2.1)),

$$q[0, \xi](\phi_\xi) \leq \alpha(a, m) + C\gamma_\xi.$$

Using the min-max principle, (3.37), (3.39), we get the following upper bound :

$$\lambda_1^N(\xi) \leq \Theta_0 + \frac{C}{m^\delta}, \quad \forall m \in [m_\delta, +\infty[.$$

Remembering the definition of Θ_0 in (2.8) and (2.13), we get $\lambda_1^N(\xi) \geq \Theta_0$. Therefore, we have, after applying Taylor's Formula to $\lambda_1^N(\cdot)$ up to the order 2 near ξ_0,

$$|\xi - \xi_0| \leq \frac{C}{m^{\delta/2}}.$$

This achieves the proof of the proposition. Let us mention also that it results now from the relation (3.38),

$$\gamma_\xi = \frac{\sqrt{a\Theta_0}}{\sqrt{m}}(1 + o(1)), \quad (m \to +\infty). \tag{3.42}$$

$\square$

The following lemma is very useful for the localization of the set $\mathrm{M}(a, m, \alpha)$ when $m \to +\infty$.

Lemma 3.11. *Given $a > 0$, there exist $m_0 > 1$ and a function $m \mapsto \epsilon(m)$ satisfying*

$$\epsilon(m) > 0, \qquad \lim_{m \to +\infty} \epsilon(m) = 0,$$

such that, if $m > m_0$, we have :

$$\gamma(a, m, \alpha; \xi) > 0, \quad \forall \xi \in] - \epsilon(m) + \xi_0, \xi_0 + \epsilon(m)[, \tag{3.43}$$

and

$$\alpha + \mu_1(a, m, \alpha; \xi) = \lambda_1 \left(\gamma(a, m, \alpha; \xi), \xi\right), \quad \forall \xi \in] - \epsilon(m) + \xi_0, \xi_0 + \epsilon(m)[. \tag{3.44}$$

Proof. Let $\xi \in \mathbb{R}_+$. Looking at the eigenvalue equation satisfied by $f_{\alpha,\xi}^{a,m}$, we get :

$$\begin{cases} -\left(f_{\alpha,\xi}^{a,m}\right)''(t) + (t - \xi)^2 f_{\alpha,\xi}^{a,m}(t) = m(\mu_1(a, m, \alpha; \xi) - a\alpha) f_{\alpha,\xi}^{a,m}, \quad t < 0, \\ \left(f_{\alpha,\xi}^{a,m}\right)'(0_-) = m\,\gamma(a, m, \alpha; \xi) f_{\alpha,\xi}^{a,m}(0). \end{cases}$$

When $|\xi - \xi_0| < \epsilon(m)$, it results from the min-max principle the existence of $m_0 > 0$ such that :

$$\forall m \geq m_0, \quad \mu_1(a, m, \alpha; \xi) \leq \frac{1}{2} a\Theta_0.$$

To obtain the above estimate, it is enough to use the function $\chi(\sqrt{m}t)\widetilde{\varphi}_\xi(t)$ as a quasi-mode (cf. (3.35)), where the function $\varphi_\xi = \varphi_{0,\xi}$ is the eigenfunction associated with $\mu^N(\xi)$ (cf. (2.4)), and $\widetilde{\varphi}_\xi$ is the even extension of φ_ξ.
Therefore, remembering that $\alpha > \Theta_0$,

$$\mu_1(a, m, \alpha; \xi) - a\alpha < 0 \quad \forall m \geq m_0,$$

and consequently we obtain (3.43). Looking now at the eigenvalue equation in $\mathbb{R}_+$:

$$\begin{cases} -\left(f_{\alpha,\xi}^{a,m}\right)''(t) + (t - \xi)^2 f_{\alpha,\xi}^{a,m}(t) = (\mu_1(a, m, \alpha; \xi) + \alpha) f_{\alpha,\xi}^{a,m}, \quad t > 0, \\ \left(f_{\alpha,\xi}^{a,m}\right)'(0_+) = \gamma(a, m, \alpha; \xi) f_{\alpha,\xi}^{a,m}(0), \end{cases}$$

with, thanks to (3.8),

$$\alpha + \mu_1(a, m, \alpha; \xi) \leq \lambda_1^D(\xi).$$

As $\gamma(a, m, \alpha; \xi) > 0$ when $m \geq m_0$, then, thanks to Lemma 2.1, we obtain formula (3.44). $\square$

In the next lemma we give a two-term asymptotics to $\alpha(a, m)$.

Lemma 3.12. *The following asymptotic expansion holds,*

$$\alpha(a,m) = \Theta_0 + 3C_1\frac{\sqrt{a\Theta_0}}{\sqrt{m}} + \mathcal{O}\left(\frac{1}{m}\right), \quad (m \to +\infty), \tag{3.45}$$

where the constant C_1 is defined in (2.14).

Remark 3.13. *We believe that $\alpha(a,m)$ has a complete asymptotic expansion in powers of $\frac{1}{\sqrt{m}}$ as $m \to +\infty$.*

Proof of Lemma 3.12. Let $\xi \in \mathrm{M}(a,m,\alpha)$ and $\gamma = \gamma(a,m,\alpha;\xi)$. It is sufficient to establish, thanks to (2.11), (2.14) and (3.42), the existence of constants $m_0, C > 0$ such that,

$$|\alpha - \Theta(\gamma)| \leq \frac{C}{m}, \quad \forall m \geq m_0. \tag{3.46}$$

Let us recall that $\mu_1(a,m,\alpha;\xi) = 0$. The definition of $\Theta(\gamma)$ (cf. (2.8)) together with (3.44) gives the following lower bound for α :

$$\Theta(\gamma) \leq \alpha.$$

We look now for an upper bound. Consider the following quasi-mode,

$$u(t) = \begin{cases} \varphi_\gamma(t), & t > 0, \\ \varphi_\gamma(0)\exp(b_m t), & t < 0, \end{cases}$$

where φ_γ is defined in (2.12) and the parameter $b_m > 0$ is to be chosen appropriately.
Let us notice that u is in the form domain of $Q[a,m,\alpha;\xi]$, and that,

$$\int_{-\infty}^0 \left|\left(e^{b_m t}\right)'\right|^2 dt = \frac{b_m}{2}, \quad \int_{-\infty}^0 e^{2b_m t}dt = \frac{1}{2b_m},$$
$$\int_{-\infty}^0 (t-\xi(\gamma))^2 e^{2b_m t}dt = \frac{1}{4b_m^2} + \frac{\xi(\gamma)(1+\xi(\gamma))}{2b_m}.$$

Hence, we obtain,

$$\begin{aligned} Q[a,m,\alpha;\xi(\gamma)](u) \quad &\leq \quad \Theta(\gamma) - \alpha + \left(\frac{b_m}{2m} - \gamma + \frac{a\alpha}{2b_m}\right)|\varphi_\gamma(0)|^2 \\ &\quad + \frac{1}{m}\left(\frac{1}{4b_m^2} + \frac{\xi(\gamma)(1+\xi(\gamma))}{2b_m}\right)|\varphi_\gamma(0)|^2. \end{aligned} \tag{3.47}$$

On the other hand, thanks to the min-max principle and the choice of α, we have

$$Q[a,m,\alpha;\xi(\gamma)](u) \geq 0.$$

Let us choose b_m in the form :

$$b_m = b_0\sqrt{m}, \qquad \text{with } b_0 \geq 0.$$

Noticing that $|\varphi_\gamma(0)|$ is bounded, thanks to (3.39), Formula (3.47) can be rewritten as :

$$\alpha \leq \Theta(\gamma) + \left(\frac{b_m}{2m} - \gamma + \frac{a\alpha}{2b_m}\right)|\varphi_\gamma(0)|^2 + \frac{C_0}{m}.$$

Having in mind (3.42), we obtain,

$$\left|\left(\frac{b_m}{2m}-\gamma+\frac{a\alpha}{2b_m}\right)-\frac{(b_0-(a\Theta_0)^{1/2})^2}{2b_0\sqrt{m}}\right|\leq\frac{C}{m}.$$

Optimizing over b_0 leads to the choice $b_0=(a\Theta_0)^{1/2}$. We get therefore the following upper bound,

$$\alpha\leq\Theta(\gamma)+\frac{C}{m},$$

and thus, we achieved the proof of the lemma. □

We give now a fine localization of the set $\mathrm{M}(a,m,\alpha)$.

Proposition 3.14. *Given $a>0$, there exist constants $C,m_0>0$ such that,*

$$\forall m\geq m_0,\quad \mathrm{M}(a,m,\alpha)\subset\left]\xi_0+\frac{b}{\sqrt{m}}-\frac{C}{m},\,\xi_0+\frac{b}{\sqrt{m}}+\frac{C}{m}\right[,\tag{3.48}$$

where the constant $b>0$ is defined by,

$$b=\sqrt{a}\frac{3C_1(1-3C_1)}{2(2-3C_1)},\tag{3.49}$$

and the constant $C_1>0$ is introduced in (2.14).

Proof. Let us take $\xi\in\mathrm{M}(a,m,\alpha)$. It is sufficient to prove the existence of constants $C>0$, $m_0>0$, independent of m and such that,

$$\left|\xi-\frac{b}{\sqrt{m}}\right|\leq\frac{C}{m}.\tag{3.50}$$

Let $\gamma=\gamma(a,m,\alpha;\xi)$. Using (3.44), we get $\alpha=\lambda_1(\gamma,\xi)$. Upon applying Taylor's formula up to the order 2 to the function $\lambda_1(\cdot,\cdot)$ near $(\gamma,\xi(\gamma))$, we get, thanks also to Proposition 3.10,

$$(\xi-\xi(\gamma))^2=\frac{1}{\xi_0}(\alpha-\Theta(\gamma))(1+o(1)),\quad \text{as } m\to+\infty.\tag{3.51}$$

This gives now, thanks to (3.46),

$$|\xi-\xi(\gamma)|\leq\frac{C}{m}.$$

Consequently, Taylor's Formula applied to the function $\lambda_1(\cdot,\cdot)$ at $(0,\xi_0)$ will give as $m\to+\infty$ the following asymptotics,

$$\begin{aligned}\alpha &= \lambda_1(\gamma,\xi)\\ &= \Theta_0+a_1\gamma+a_2'\gamma^2+b_1\gamma(\xi-\xi_0)+c_1(\xi-\xi_0)^2\\ &\quad+\mathcal{O}(\gamma^3+|\xi-\xi_0|^3),\end{aligned}\tag{3.52}$$

where the coefficients a_1,a_2',b_1,c_1 are defined by,

$$a_1=(\partial_\gamma\lambda_1)(0,\xi_0)=\Theta'(0),\quad a_2'=\frac{1}{2}\left(\partial_\gamma^2\lambda_1\right)(0,\xi_0),$$
$$b_1=\frac{1}{2}(\partial_\gamma\partial_\xi\lambda_1)(0,\xi_0),\quad c_1=\xi_0.$$

One has also, thanks to Taylor's Formula (applied to the function $\Theta(\gamma)$) and to (2.10) ,

$$\begin{aligned}\Theta(\gamma) &= \Theta_0 + a_1\gamma + a_2\gamma^2 + \mathcal{O}(\gamma^3), &(3.53)\\ \xi(\gamma) &= \xi_0 + \frac{a_1}{2\xi_0}\gamma + \mathcal{O}(\gamma^2), &(3.54)\end{aligned}$$

where the coefficients a_1, a_2 are defined by,

$$a_1 = \Theta'(0), \quad a_2 = \frac{1}{2}\Theta''(0).$$

By writing $(\xi - \xi(\gamma))^2 = (\xi - \xi_0)^2 + (\xi - \xi(\gamma))^2 - 2(\xi - \xi_0)(\xi - \xi(\gamma))$, we obtain thanks to (3.32) and (3.54),

$$(\xi - \xi(\gamma))^2 = (\xi - \xi_0)^2 + (\xi(\gamma) - \xi_0)^2 - \frac{a_1}{\xi_0}\gamma(\xi - \xi_0) + o(\gamma^2). \tag{3.55}$$

Using the formulas of differentiation in (2.6) and (2.7), we get,

$$\begin{aligned}&b_1 = -\frac{1}{2}a_1^2,\\ &\Theta''(\gamma) = \left(\partial_\gamma^2\lambda_1\right)(\gamma, \xi(\gamma)) + \xi'(\gamma)\left(\partial_\gamma\partial_\xi\lambda_1\right)(\gamma, \xi(\gamma)),\\ &a_2' = a_2 + \frac{a_1^3}{4\xi_0}.\end{aligned} \tag{3.56}$$

Solving the equation (3.51), we get finally,

$$\xi - \xi_0 = \frac{a_1(1 - a_1)}{4\xi_0\left(1 - \frac{a_1}{2}\right)}\gamma(1 + o(1)), \quad (m \to +\infty).$$

Recalling (3.42), we obtain (3.50). Finally, it is proved in [17, (2.13)] that $a_1 < 1$ (and hence $b > 0$). ☐

Remark 3.15. *The following numerical estimate was obtained in [10, Formula (2.125)] :*

$$0.858 \le 3C_1 \le 0.888.$$

This shows that $b > 0$.

The asymptotic behavior of $\alpha(a, m)$ as $m \to +\infty$ permits one to prove the existence of a unique minimum of the function $\mu_1(a, m, \alpha; \cdot)$ for large values of the parameter m.

Theorem 3.16. *Given $a > 0$, there exists $m_0 > 1$ such that, for any $m \ge m_0$, the function $\xi \mapsto \mu_1(a, m, \alpha(a, m); \cdot)$ has a unique non-degenerate positive minimum denoted by $\xi(a, m)$.*

Proof. Let us take, thanks to Theorems 3.5 and 3.14, a critical point

$$\xi \in \left]\xi_0 + \frac{b}{\sqrt{m}} - \frac{C}{m}, \xi_0 + \frac{b}{\sqrt{m}} + \frac{C}{m}\right[.$$

It is sufficient to prove that,

$$\partial_\xi^2\mu_1(a, m, \alpha; \xi) > 0. \tag{3.57}$$

Notice that, Formula (3.10) gives,

$$\partial_\xi^2 \mu^{(1)}(a,m,\alpha;\xi) = 2(m-1)\left[\gamma'(a,m,\alpha;\xi)\gamma(a,m,\alpha;\xi) + \frac{1}{m}\xi\right] |f_{\alpha,\xi}^{a,m}(0)|^2. \tag{3.58}$$

Differentiating both sides of (3.44), we get, thanks to (2.6),

$$\gamma'(a,m,\alpha;\xi) = |\gamma(a,m,\alpha;\xi)|^2 - \xi^2 + \alpha + \mu_1(a,m,\alpha;\xi). \tag{3.59}$$

Notice that, thanks to (3.42) and (3.45),

$$|\gamma(a,m,\alpha;\xi)|^2 - \xi^2 + \alpha = \left(a_1 - 2\frac{b}{\sqrt{a}\,\xi_0}\right)\gamma(1+o(1)), \quad (m \to +\infty),$$

with $a_1 = 3C_1$. Since, $0 < a_1 < 1$ (cf. [17, (2.13)]), we get, thanks to (3.49),

$$a_1 - 2\frac{b}{\sqrt{a}\,\xi_0} > 0.$$

We have therefore achieved the proof of the theorem. $\square$

Using the regularity result in Theorem 3.1, one gets immediately the following consequence of Theorems 3.5 and 3.16.

Theorem 3.17. *Given $a > 0$, there exists $m_0 > 1$ such that :*

$$\forall m \geq m_0, \quad \exists\, \epsilon_0(m) > 0,$$

and if $\widehat{\alpha}$ satisfies :

$$|\widehat{\alpha} - \alpha(a,m)| \leq \epsilon_0(m),$$

then the function

$$\xi \mapsto \mu_1(a,m,\widehat{\alpha};\xi)$$

has a unique positive non-degenerate minimum which we denote by $\widehat{\xi}(a,m)$.

4 Analysis of a 'refined' family of model operators

4.1 Notation and main theorem

For the analysis of 'curvature effects' (Theorem 1.3), we need to introduce a refined family of model operators. Let us consider

$$a, m, \widehat{\alpha}, h \in \mathbb{R}_+, \quad \beta, \xi \in \mathbb{R} \quad \text{and} \quad \delta \in]\frac{1}{4}, \frac{1}{2}[. \tag{4.1}$$

We assume further the following condition on β, h and δ,

$$\beta h^\delta < 1.$$

We consider the quadratic form :

$$H_0^1(]-h^{\delta-1/2}, h^{\delta-1/2}[) \ni u \mapsto q_{h,\beta,\xi}^{a,m,\widehat{\alpha}}(u),$$

defined by

$$q_{h,\beta,\xi}^{a,m,\widehat{\alpha}}(u) = \int_0^{h^{\delta-1/2}} \Big[|u'(t)|^2 + (1+2\beta h^{1/2}t)\left|\left(t-\xi-\beta h^{1/2}\frac{t^2}{2}\right)u(t)\right|^2 -\widehat{\alpha}|u(t)|^2\Big](1-\beta h^{1/2}t)dt + \frac{1}{m}\int_{-h^{\delta-1/2}}^0 \Big[|u'(t)|^2 + (1+2\beta h^{1/2}t)\left|\left(t-\xi-\beta h^{1/2}\frac{t^2}{2}\right)u(t)\right|^2 + a\widehat{\alpha}|u(t)|^2\Big](1-\beta h^{1/2}t)dt. \tag{4.2}$$

By Friedrichs' Theorem, we associate to the quadratic form $q_{h,\beta,\xi}^{a,m,\widehat{\alpha}}$ a non-bounded self-adjoint operator $\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}}$ on the space

$$L^2(]-h^{\delta-1/2},h^{\delta-1/2}[;(1-\beta h^{1/2}t)dt).$$

The domain of $\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}}$ is defined by :

$$\begin{aligned} D(\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}}) &= \{u\in H_0^1(]-h^{\delta-1/2},h^{\delta-1/2}[);\quad u_{|_{]-h^{\delta-1/2},0[}}\in H^2(]-h^{\delta-1/2},0[),\\ &\qquad u_{|_{]0,h^{\delta-1/2}[}}\in H^2(]0,h^{\delta-1/2}[);\quad u'(0_+)=\frac{1}{m}u'(0_-)\}. \end{aligned} \tag{4.3}$$

For $u\in D(\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}})$, we have :

$$\left(\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}}u\right)(t) = \begin{cases} (H_{h,\beta,\xi}u-\widehat{\alpha}u)(t), & \text{if}\quad t<0,\\ \left(\frac{1}{m}H_{h,\beta,\xi}u+a\widehat{\alpha}u\right)(t), & \text{if}\quad t>0, \end{cases} \tag{4.4}$$

where $H_{h,\beta,\xi}$ is the differential operator :

$$\begin{aligned} H_{h,\beta,\xi} &= -\partial_t^2+(t-\xi)^2\\ &\quad +\beta h^{1/2}(1-\beta h^{1/2}t)^{-1}\partial_t + 2\beta h^{1/2}t\left(t-\xi-\beta h^{1/2}\frac{t^2}{2}\right)^2\\ &\quad -\beta h^{1/2}t^2(t-\xi)+\beta^2 h\frac{t^4}{4}. \end{aligned} \tag{4.5}$$

We denote by $\mu_j(\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}})$ the increasing sequence of eigenvalues of $\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}}$. We are interested in finding a lower bound of

$$\inf_{\xi\in\mathbb{R}}\mu_1\left(\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}}\right).$$

We shall always work under the following general hypothesis,

$$m\geq m_0, \tag{4.6}$$

where the constant $m_0>1$ fills the assumption of Theorem 3.17. We write,

$$\alpha=\alpha(a,m). \tag{4.7}$$

We suppose also that (cf. Theorem 3.17) :

$$|\widehat{\alpha}-\alpha|\leq\epsilon_0(m).$$

We denote then by $\widehat{\eta}$ and η the unique numbers defined by Theorem 3.17 such that :

$$\widehat{\eta} \in \mathrm{M}(a,m,\widehat{\alpha}), \quad \eta \in \mathrm{M}(a,m,\alpha), \tag{4.8}$$

and we recall that $\widehat{\eta},\eta > 0$. Finally, we denote by :

$$\widehat{f} = f^{a,m}_{\widehat{\alpha},\widehat{\eta}}, \quad f = f^{a,m}_{\alpha,\eta} \tag{4.9}$$

the positive eigenfunctions (and normalized for the L^2-norm in $\mathbb{R}$) associated to $\mu_1(a,m,\widehat{\alpha};\widehat{\eta})$ and $\mu_1(a,m,\alpha;\eta)$ respectively.
We define the following functions :

$$\widetilde{\mathcal{C}}_1(a,m) = \int_{\mathbb{R}_+} (t-\eta)^3 |f(t)|^2 dt + \frac{1}{m}\int_{\mathbb{R}_-} (t-\eta)^3|f(t)|^2 dt - \frac{1}{2}\left(1-\frac{1}{m}\right)|f(0)|^2, \tag{4.10}$$

$$b_1(a,m) = \int_{\mathbb{R}_+} |f(t)|^2 dt - a\int_{\mathbb{R}_-} |f(t)|^2 dt. \tag{4.11}$$

Notice that, thanks to (3.22) and the asymptotic behavior as $m \to +\infty$, the constant $\widetilde{\mathcal{C}}_1(a,m)$ is negative for large values of the parameter m.
Our aim in this section is to prove the following theorem.

Theorem 4.1. *Given $a > 0$ and $m \geq m_0$, then for every $M > 0$, there exist constants $C, \epsilon, h_0 > 0$ such that :*

$$\forall \beta \in]-M, M[, \quad \forall \xi \in \mathbb{R}, \quad \forall \widehat{\alpha} \in [-\epsilon+\alpha, \alpha+\epsilon], \quad \forall h \in]0, h_0],$$

we have :

$$\mu_1(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}) \geq b_1(a,m)(\widehat{\alpha}-\alpha) + \widetilde{\mathcal{C}}_1(a,m)\beta h^{1/2} - C\left[|\widehat{\alpha}-\alpha|^2 + h^{\delta+1/4}\right]. \tag{4.12}$$

Let us mention that in all the estimates of this section, we do not seek to control the constants uniformly with respect to the parameter m.

4.2 A first order approximation of $\mu_1(a,m,\widehat{\alpha};\widehat{\eta})$

We construct a first order approximation of $\mu_1(a,m,\widehat{\alpha};\widehat{\eta})$ by the help of an approximate eigenfunction. We recall first the definition of the 'regularized resolvent'. Given $a,m,\beta > 0$ and $\xi \in \mathbb{R}$, the regularized resolvent $\mathcal{R}[a,m,\beta;\xi]$ is the bounded linear operator on $L^2(\mathbb{R})$ defined by,

$$\mathcal{R}[a,m,\beta;\xi]u = \begin{cases} \left(H[a,m,\beta;\xi] - \mu_1(a,m,\beta;\xi)\right)^{-1} u, & \text{if} \quad u \perp f^{a,m}_{\beta,\xi}, \\ 0, \quad \text{if} \quad u \in \mathbb{R}\cdot f^{a,m}_{\beta,\xi}. \end{cases} \tag{4.13}$$

Lemma 4.2. *There exist constants $C, \epsilon > 0$ such that, if $|\widehat{\alpha}-\alpha| \leq \epsilon$, we have :*

$$\left|\mu_1(a,m,\widehat{\alpha};\widehat{\eta}) - \mu_1(a,m,\alpha;\widehat{\eta}) - \widehat{b}_1(a,m)(\widehat{\alpha}-\alpha)\right| \leq C|\widehat{\alpha}-\alpha|^2, \tag{4.14}$$

where

$$\widehat{b}_1(a,m) = \int_{\mathbb{R}_+} |f^{a,m}_{\alpha,\widehat{\eta}}(t)|^2 dt - a\int_{\mathbb{R}_-} |f^{a,m}_{\alpha,\widehat{\eta}}(t)|^2 dt.$$

Preuve. Let $\tau = \widehat{\alpha} - \alpha$. We look for $b_0, b_1 \in \mathbb{R}$ and $u_0, u_1 \in L^2(\mathbb{R})$ such that,

$$(H[a,m,\widehat{\alpha};\widehat{\eta}] - (b_0 + b_1\tau))(u_0 + \tau u_1) \sim 0 \quad \text{in } \mathbb{R}.$$

Let us write,

$$\begin{aligned}&(H[a,m,\widehat{\alpha};\widehat{\eta}] - (b_0 + b_1\tau))(u_0 + \tau u_1)\\ &= (H[a,m,\alpha;\widehat{\eta}] - b_0)\,u_0 + \tau\left\{(H[a,m,\alpha;\widehat{\eta}] - b_0)\,u_1 - (b_1 + \zeta)u_0\right\} + \tau^2 R,\end{aligned}$$

where

$$\zeta(t) = \begin{cases} -1; & \text{if} \quad t > 0,\\ a; & \text{if} \quad t < 0,\end{cases}$$

and

$$R = \begin{cases} (b_1 - 1)u_1, & \text{if} \quad t > 0,\\ (b_1 + a)u_1, & \text{if} \quad t < 0.\end{cases}$$

We choose b_0, b_1, u_0, u_1 in the following way :

$$\begin{aligned}&b_0 = \mu^{(1)}(a,m,\alpha;\widehat{\eta}), \quad u_0 = f^{a,m}_{\alpha,\widehat{\eta}},\\ &b_1 = \int_{\mathbb{R}_+} |u_0|^2 dt - a\int_{\mathbb{R}_-} |u_0|^2 dt, \quad u_1 = \mathcal{R}[a,m,\alpha;\widehat{\eta}]g,\end{aligned}$$

where the function $g \in u_0^\perp$ is defined by,

$$g = \begin{cases} (b_1 - 1)u_0; & if \quad t > 0,\\ (b_1 + a)u_0; & if \quad t < 0.\end{cases}$$

Notice that $u := u_0 + \tau u_1$ is in the domain of the operator $H[a,m,\widehat{\alpha};\widehat{\eta}]$. Therefore, the constructions above together with the spectral theorem yields the existence of an eigenvalue $\widetilde{\mu}$ of $H[a,m,\widehat{\alpha};\widehat{\eta}]$ that satisfies :

$$|\widetilde{\mu} - \mu_1(a,m,\alpha;\widehat{\eta}) - b_1\tau| \leq C\tau^2.$$

By comparing the quadratic forms $Q[a,m,\widehat{\alpha};\widehat{\eta}]$ and $Q[a,m,\alpha;\widehat{\eta}]$, we obtain, thanks to the min-max principle,

$$|\mu_2(a,m,\widehat{\alpha};\widehat{\eta}) - \mu_2(a,m,\alpha;\widehat{\eta})| \leq C\tau.$$

Therefore, the only possible choice of $\widetilde{\mu}$ is $\widetilde{\mu} = \mu_1(a,m,\widehat{\alpha};\widehat{\eta})$. This achieves the proof of the lemma. □

We determine in the next lemma a useful 'key' estimate of $|\widehat{\eta} - \eta|$.

Lemma 4.3. *Let α be as in (4.7). There exist constants $C, \epsilon > 0$ such that, if $|\widehat{\alpha} - \alpha| \leq \epsilon$, then we have :*

$$|\widehat{\eta} - \eta| \leq C|\widehat{\alpha} - \alpha|. \tag{4.15}$$

Here $\eta, \widehat{\eta}$ are introduced in (4.8).

Preuve. We denote by :

$$\widehat{\gamma} := \gamma(a,m,\widehat{\alpha};\widehat{\eta}),\quad \gamma := \gamma(a,m,\alpha;\eta),$$

where $\gamma(\cdot)$ is defined in (3.6). As $\widehat{\eta}$ (respectively η) is a critical point of the function $\mu_1(a,m,\widehat{\alpha};\cdot)$ (respectively of $\mu_1(a,m,\alpha;\cdot)$, we get, thanks to formula (3.10) :

$$\widehat{\gamma}^2 = \frac{a+1}{m-1}\widehat{\alpha} - \frac{1}{m}\widehat{\eta}^2,\quad \gamma^2 = \frac{a+1}{m-1}\alpha - \frac{1}{m}\eta^2,$$

and consequently,

$$\widehat{\gamma}^2 - \gamma^2 = \frac{a+1}{m-1}(\widehat{\alpha}-\alpha) - \frac{1}{m}(\widehat{\eta}^2 - \eta^2). \tag{4.16}$$

Writing Taylor's formula up to the order 1 of the function $\xi \mapsto \gamma(a,m,\alpha;\xi)$ near η, we obtain,

$$\widehat{\gamma} = \gamma + c_1(\widehat{\eta}-\eta) + \mathcal{O}(|\widehat{\eta}-\eta|^2),$$

where $c_1 = \gamma'(\eta)$ is given by, thanks to (3.59),

$$c_1 = \gamma^2 - \eta^2 + \alpha.$$

Substituting in (4.16), we obtain :

$$\left(c_1\gamma + \frac{\eta}{m}\right)(\widehat{\eta}-\eta) + \mathcal{O}(|\widehat{\eta}-\eta|^2) = \mathcal{O}(|\widehat{\alpha}-\alpha|).$$

Since η is a non-degenerate minimum of $\mu_1(a,m,\alpha;\cdot)$, then, thanks to (3.58),

$$c_1\gamma + \frac{\eta}{m} > 0.$$

This achieves the proof of the lemma. $\square$

Lemmas 4.2 and 4.3 give now the following theorem, thanks to the regularity in Theorem 3.1.

Theorem 4.4. *There exist constants $C, \epsilon > 0$ such that, if $|\widehat{\alpha}-\alpha| \leq \epsilon$, we have,*

$$|\mu_1(a,m,\widehat{\alpha};\widehat{\eta}) - b_1(a,m)(\widehat{\alpha}-\alpha)| \leq C|\widehat{\alpha}-\alpha|^2. \tag{4.17}$$

4.3 A lower bound of $\mu_1(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi})$

We start by the following 'rough' localization of the spectrum of $\mathcal{H}^{a,m,n}_{h,\beta,\xi}$.

Proposition 4.5. *Given $a, m, M > 0$, there exist constants $C, h_0 > 0$ such that*

$$\forall \beta \in]-M, M[,\quad \forall \xi \in \mathbb{R},\quad \forall h \in]0, h_0],$$

we have,

$$\left|\mu_j(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}) - \mu_j(\mathcal{H}^{a,m,\widehat{\alpha}}_{0,\xi})\right| \leq Ch^{2\delta-1/2}\left(1 + \mu_j(\mathcal{H}^{a,m,\widehat{\alpha}}_{0,\xi})\right), \tag{4.18}$$

where the operator $\mathcal{H}^{a,m,\widehat{\alpha}}_{0,\xi}$ is defined by :

$$D(\mathcal{H}^{a,m,\widehat{\alpha}}_{0,\xi}) := D(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}),\quad H^{a,m,\widehat{\alpha}}_{0,\xi} = H[a,m,\widehat{\alpha};\xi].$$

The estimate (4.18) is obtained by first comparing the corresponding quadratic forms and then by applying the min-max principle. Notice that the min-max principle gives also, thanks to the inclusion of the form domains,

$$\forall j \in \mathbb{N}, \quad \mu_j(\mathcal{H}_{0,\xi}^{a,m,\widehat{\alpha}}) \geq \mu_j(a, m, \widehat{\alpha}; \xi), \tag{4.19}$$

where $\mu_j(a, m, \widehat{\alpha}; \xi)$ is the increasing sequence of eigenvalues of the operator $H[a, m, \widehat{\alpha}; \xi]$.
Since $\widehat{\eta}$ is a non-degenerate minimum of $\mu^{(1)}(a, m, \widehat{\alpha}; \cdot)$, we get the following lower bound of $\mu_1(\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}})$, when ξ is not very near $\widehat{\eta}$.

Proposition 4.6. *Given $a, m, M > 0$, there exist constants $C, \zeta, h_0 > 0$ such that :*

$$\forall \beta \in]-M, M[, \quad \forall \xi \; : \; |\xi - \widehat{\eta}| \geq \zeta h^{\delta - 1/4}, \quad \forall h \in]0, h_0], \tag{4.20}$$

we have,

$$\mu_1(\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}}) \geq \mu_1(a, m, \widehat{\alpha}; \widehat{\eta}) + Ch^{2\delta - 1/2}. \tag{4.21}$$

Proof. Recalling the localization of the spectrum in (4.18) and (4.19), it is sufficient to prove the existence of $\zeta > 0$ and h_0 such that, under the hypothesis (4.20), we have,

$$\mu_1(a, m, \widehat{\alpha}; \xi) \geq \mu_1(a, m, \widehat{\alpha}; \widehat{\eta}) + h^{2\delta - 1/2}.$$

Using Taylor's formula up to the order 2, we get constants $\theta, C_0 > 0$ such that :

$$\begin{aligned} \mu_1(a, m, \widehat{\alpha}; \xi) \quad &= \quad \mu_1(a, m, \widehat{\alpha}; \widehat{\eta}) + \frac{|\xi - \widehat{\eta}|^2}{2} \partial_\xi^2 \mu_1(a, m, \widehat{\alpha}; \xi)_{\xi = \widehat{\eta}} \\ &\quad - C_0 |\xi - \widehat{\eta}|^3, \quad \forall \xi \in]\widehat{\eta} - \theta, \widehat{\eta} + \theta[. \end{aligned} \tag{4.22}$$

The constant C_0 is uniform with respect to $\widehat{\alpha}$, thanks to the regularity in Theorem 3.1.
Since $\partial_\xi^2 \mu^{(1)}(a, m, \widehat{\alpha}; \xi)_{\xi = \widehat{\eta}} > 0$ (cf. Theorem 3.17), there exist constants $C_0' > 0$ such that we can rewrite (4.22) as :

$$\mu_1(a, m, \widehat{\alpha}; \xi) \geq \mu_1(a, m, \widehat{\alpha}; \widehat{\eta}) + C_0' |\xi - \widehat{\eta}|^2, \quad \forall \xi \in]\widehat{\eta} - \theta, \widehat{\eta} + \theta[.$$

We choose $\zeta > 0$ in such a way that $C_0' \zeta > 1$. We then obtain obtain for $\zeta h^{\delta - 1/2} \leq |\xi - \widehat{\eta}| < \theta$,

$$\mu_1(a, m, \widehat{\alpha}; \xi) \geq \mu_1(a, m, \widehat{\alpha}; \widehat{\eta}) + h^{2\delta - 1/2}.$$

If $|\xi - \widehat{\eta}| \geq \theta$, there exists, thanks to the variation of $\mu_1(a, m, \widehat{\alpha}; \cdot)$, a constant $\epsilon_\theta > 0$ such that,

$$\mu_1(a, m, \widehat{\alpha}; \xi) \geq \mu_1(a, m, \widehat{\alpha}; \widehat{\eta}) + \epsilon_\theta.$$

It is sufficient now to choose h_0 in such a way that $h_0^{2\delta - 1/2} < \epsilon_\theta$. □

We suppose now that $|\xi - \widehat{\eta}| \leq \zeta h^{2\delta - 1/2}$. In this case we follow the general technique initiated in [24, Section 11] to construct a formal asymptotic expansion for $\mu_1(\mathcal{H}_{h,\beta,\xi}^{a,m,\widehat{\alpha}})$ in powers of $(\xi - \widehat{\eta})$.
We look for a formal solution $\left(\mu, f_{h,\beta,\xi}^{a,m,\widehat{\alpha}}\right)$ of the following eigenvalue problem[6],

$$\begin{cases} \left(H_{h,\beta,\xi} - \widehat{\alpha}\right) f_{h,\beta,\xi}^{a,m,\widehat{\alpha}} \sim \mu f_{h,\beta,\xi}^{a,m,\widehat{\alpha}}, \quad \text{in } \mathbb{R}_+, \\ \left(\frac{1}{m} H_{h,\beta,\xi} + a\widehat{\alpha}\right) f_{h,\beta,\xi}^{a,m,\widehat{\alpha}} \sim \mu f_{h,\beta,\xi}^{a,m,\widehat{\alpha}}, \quad \text{in } \mathbb{R}_-, \\ (f_{h,\beta,\xi}^{a,m,\widehat{\alpha}})'(0_+) = \frac{1}{m} (f_{h,\beta,\xi}^{a,m,\widehat{\alpha}})'(0_-) \text{ in } \mathbb{R}, \end{cases} \tag{4.23}$$

[6]For two functions $f(h), g(h)$, we use the notation $f \sim g$ if $\lim_{h \to 0}([f(h)]/[g(h)] = 1$.

in the form :

$$\mu = d_0 + d_1 (\xi - \widehat{\eta}) + d_2 (\xi - \widehat{\eta})^2 + d_3 h^{1/2}, \tag{4.24}$$
$$f_{h,\beta,\xi}^{a,m,\widehat{\alpha}} = u_0 + (\xi - \widehat{\eta})\, u_1 + (\xi - \widehat{\eta})^2\, u_2 + h^{1/2} u_3, \tag{4.25}$$

where the coefficients d_0, d_1, d_2, d_3 and the functions u_0, u_1, u_2, u_3 are to be determined. We expand the operator $H_{h,\beta,\xi}$ in powers of $(\xi - \xi(\widehat{\eta}))$ and then we identify the coefficients in (4.23) of the terms of orders $(\xi - \xi(\widehat{\eta}))^j$ $(j = 0, 1, 2)$ and $h^{1/2}$. We then obtain the following 'leading order' equations,

$$(H[a, m, \widehat{\alpha}; \widehat{\eta}] - d_0)\, u_0 = 0, \tag{4.26}$$
$$(H[a, m, \widehat{\alpha}; \widehat{\eta}] - d_0)\, u_1 = g_1, \tag{4.27}$$
$$(H[a, m, \widehat{\alpha}; \widehat{\eta}] - d_0)\, u_2 = g_2, \tag{4.28}$$
$$(H[a, m, \widehat{\alpha}; \widehat{\eta}] - d_0)\, u_3 = g_3, \tag{4.29}$$

where the function g_1, g_2 and g_3 are defined by :

$$g_1(t) = \begin{cases} 2(t - \widehat{\eta})u_0, & \text{if} \quad t > 0, \\ \frac{2}{m}(t - \widehat{\eta})u_0, & \text{if} \quad t < 0, \end{cases} \tag{4.30}$$

$$g_2(t) = \begin{cases} 2\left[(t - \widehat{\eta})u_1 + d_1\right] + (d_2 - 1)u_0, & \text{if} \quad t > 0, \\ 2\left[\frac{1}{m}(t - \widehat{\eta}) + d_1\right] u_1 + (d_2 - \frac{1}{m})u_0, & \text{if} \quad t < 0, \end{cases} \tag{4.31}$$

and

$$g_3(t) = \begin{cases} -\left[\beta\left(\partial_t + (t - \widehat{\eta})^3 - \widehat{\eta}^2\,(t - \widehat{\eta})\right) - d_3\right] u_0, & \text{if} \quad t > 0, \\ -\left[\frac{1}{m}\beta\left(\partial_t + (t - \widehat{\eta})^3 - \widehat{\eta}^2\,(t - \widehat{\eta})\right) - d_3\right] u_0, & \text{if} \quad t < 0. \end{cases} \tag{4.32}$$

We recall that (cf. (3.21)) :

$$\int_{\mathbb{R}_+} (t - \widehat{\eta})|\widehat{f}(t)|^2 dt + \frac{1}{m}\int_{\mathbb{R}_-} (t - \widehat{\eta})|\widehat{f}(t)|^2 dt = 0. \tag{4.33}$$

Then, thanks to (4.33), we can choose d_0, d_1, d_2, d_3 and the functions u_0, u_1, u_2, u_3 in the following way :

$$d_0 = \mu_1(a, m, \widehat{\alpha}; \widehat{\eta}), \quad u_0 = \widehat{f}, \tag{4.34}$$
$$d_1 = 0, \quad u_1 = 2\mathcal{R}[a, m, \widehat{\alpha}; \widehat{\eta}]g_1, \tag{4.35}$$
$$d_2 = \int_{\mathbb{R}_+} (u_0 - 2(t - \widehat{\eta})u_1)\, u_0\, dt + \frac{1}{m}\int_{\mathbb{R}_-} (u_0 - 2(t - \widehat{\eta}))\, u_0\, dt, \tag{4.36}$$
$$u_2 = 2\mathcal{R}[a, m, \widehat{\alpha}; \widehat{\eta}]g_2, \tag{4.37}$$
$$d_3 = \beta\left(\int_{\mathbb{R}_+} u_0 \cdot \left(\partial_t + (t - \widehat{\eta})^3\right) u_0\, dt + \frac{1}{m}\int_{\mathbb{R}_-} u_0 \cdot \left(\partial_t + (t - \widehat{\eta})^3\right) u_0\, dt\right), \tag{4.38}$$
$$u_3 = \beta\mathcal{R}[a, m, \widehat{\alpha}; \widehat{\eta}]g_3. \tag{4.39}$$

An integration by parts yields :

$$d_3 = \beta\,\widehat{\mathcal{C}}_1(a, m), \tag{4.40}$$

where

$$\widehat{\mathcal{C}}_1(a, m) = \int_{\mathbb{R}_+} (t - \widehat{\eta})^3|\widehat{f}(t)|^2 dt + \frac{1}{m}\int_{\mathbb{R}_-} (t - \widehat{\eta})^3|\widehat{f}(t)|^2 dt + \frac{1}{2}\left(1 - \frac{1}{m}\right)|\widehat{f}(0)|^2. \tag{4.41}$$

We define now the following quasi-mode :

$$\widehat{f}_h(t) = \chi\left(h^{-\delta+1/2}t\right) f^{a,m,\widehat{\alpha}}_{h,\beta,\xi}(t), \tag{4.42}$$

where χ is a cut-off function supported in $]-1,1[$. Since the function $\widehat{f}$ decays exponentially at infinity, we get :

$$\left|\|\widehat{f}_h\|^2_{L^2(\mathbb{R}^2)} - 1\right| \leq C[|\xi-\widehat{\eta}| + h^{1/2}], \quad \forall h \in]0,h_0].$$

We have also, thanks to the formal calculus presented above ((4.23)-(4.39)),

$$\begin{aligned}&\left|\left(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi} - [d_0 + d_2(\xi-\widehat{\eta})^2 + d_3 h^{1/2}]\right)\widehat{f}_h\right| \\ &\leq C[h^{1/2}|\xi-\widehat{\eta}| + h^{1/2+\delta}], \quad \forall h \in]0,h_0].\end{aligned}$$

By the spectral Theorem, we get an eigenvalue $\lambda(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi})$ of $\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}$ such that :

$$\begin{aligned}&\left|\lambda(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}) - [d_0 + d_2(\xi-\widehat{\eta})^2 + d_3 h^{1/2}]\right| \\ &\leq C[h^{1/2}|\xi-\widehat{\eta}| + h^{1/2+\delta}], \quad \forall h \in]0,h_0].\end{aligned}$$

The localization of the spectrum of $\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}$ in Proposition 4.5 together with the lower bound (4.19) shows that the only possible choice of $\lambda(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi})$ is

$$\lambda(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}) = \mu_1(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}).$$

Therefore, we have proved the following lemma.

Lemma 4.7. *Given $a,m,M>0$, there exist constants $C,h_0>0$ such that,*

$$\forall \beta \in]-M,M[, \quad \forall \xi \text{ s.t. } \quad |\xi-\widehat{\eta}| \leq \zeta h^{\delta-1/4}, \quad \forall h \in]0,h_0],$$

we have :

$$\left|\mu_1(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}) - [d_0 + d_2(\xi-\widehat{\eta})^2 + d_3 h^{1/2}]\right| \leq C h^{1/4+\delta}, \quad \forall h \in]0,h_0].$$

The next lemma permits to deduce that $d_2>0$.

Lemma 4.8. *Under the hypothesis of Lemma 4.7, we have :*

$$d_2 = \frac{1}{2}\partial^2_\xi \mu_1(a,m,\widehat{\alpha};\xi)_{|\xi=\widehat{\eta}}.$$

Proof. Notice that, by construction of $f^{a,m,\widehat{\alpha}}_{h,\beta,\xi}$, one has,

$$\left\|\left(H[a,m,\widehat{\alpha};\xi] - [d_0 + d_2(\xi-\widehat{\eta})^2]\right) f^{a,m,\widehat{\alpha}}_{h,\beta,\xi}\right\|_{L^2(\mathbb{R}^2)} \leq C|\xi-\widehat{\eta}|^2.$$

By the spectral theorem, we get,

$$\left|\mu_1(a,m,\widehat{\alpha};\xi) - [d_0 + d_2(\xi-\widehat{\eta})^2]\right| \leq C|\xi-\widehat{\eta}|^2.$$

Comparing the above expansion with that obtained after writing Taylor's Formula for $\mu_1(a,m,\widehat{\alpha};\xi)$ up to the order 2, one gets the result of the lemma. $\square$

Using Proposition 4.6 and Lemmas 4.7 and 4.8, we get finally the following theorem.

Theorem 4.9. *Under the hypothesis of Theorem 4.1, given $a, m, M > 0$, there exist constants $C, h_0 > 0$ such that :*

$$\forall \beta \in]-M, M[, \quad \forall \xi \in \mathbb{R}, \quad \forall h \in]0, h_0],$$

we have :

$$\mu_1(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}) \geq \mu_1(a, m, \widehat{\alpha}; \widehat{\eta}) + \widehat{\mathcal{C}}_1(a, m)h^{1/2} - Ch^{1/4+\delta}. \tag{4.43}$$

Using Theorem 4.9, the regularity of the eigenfunction in Theorem 3.1, and Lemmas 4.2 and 4.3, we achieve the proof of Theorem 4.1.

5 Estimates for the bottom of the spectrum

Let us denote by $\mathcal{P}[\kappa, H]$ the self-adjoint operator associated to the quadratic form (1.5) by Friedrichs' theorem. In this section, we estimate the bottom of the spectrum of $\mathcal{P}[\kappa, H]$ in the regime $\kappa, H \to +\infty$ and we prove Theorem 1.1.
We introduce the following parameter,

$$\varepsilon = \frac{1}{\sqrt{\kappa H}}, \tag{5.1}$$

which will be small in our analysis.

We start with a 'rough' estimate of $\mu^{(1)}(\kappa, H)$ which gives an alternative characterization of the upper critical field $H_{C_3}(a, \widetilde{m}; \kappa)$.

Proposition 5.1. *Given $a, m > 0$, there exist constants $C, \varepsilon_0 > 0$ such that, when $\varepsilon \in]0, \varepsilon_0]$, we have,*

$$\begin{aligned} -C\varepsilon + \min\left(\Theta_0 - \frac{\kappa}{H}, \frac{1}{m}\Theta_0 + a\frac{\kappa}{H}\right) \quad &\leq \quad \varepsilon^2\mu^{(1)}(\kappa, H) \\ &\leq \quad \min\left(1 - \frac{\kappa}{H}, \frac{1}{m} + a\frac{\kappa}{H}\right) + C\varepsilon. \end{aligned} \tag{5.2}$$

Moreover, there exists a constant $\kappa_0 > 0$ such that, if $\kappa \geq \kappa_0$, then,

$$H_{C_3}(a, m; \kappa) = \min\{H > 0; \quad \mu^{(1)}(\kappa, H) = 0\}. \tag{5.3}$$

Proof.
Using the min-max principle, we get,

$$\begin{aligned} &\min\left(\varepsilon^2\mu^N(\varepsilon; \Omega) - \frac{\kappa}{H}, \frac{\varepsilon^2}{m}\mu^N(\varepsilon; \Omega^c) + a\frac{\kappa}{H}\right) \\ &\leq \varepsilon^2\mu^{(1)}(\kappa, H) \\ &\leq \min\left(\varepsilon^2\mu^D(\varepsilon; \Omega) - \frac{\kappa}{H}, \frac{\varepsilon^2}{m}\mu^D(\varepsilon; \Omega^c) + a\frac{\kappa}{H}\right), \end{aligned} \tag{5.4}$$

where $\mu^N(\varepsilon; \cdot)$, $\mu^D(\varepsilon; \cdot)$ are defined in (2.26). Estimate (5.2) follows now from Proposition 2.4. We prove now (5.3). We define,

$$H_*(\kappa) = \min\{H > 0; \quad \mu^{(1)}(\kappa, H) = 0\}. \tag{5.5}$$

By definition, $H_{C_3}(a, m; \kappa) \leq H_*(\kappa)$. By (5.4), $H_*(\kappa)$ has the order of κ (as $\kappa \to +\infty$). Suppose by contradiction that there exist $H \in]0, H_*(\kappa)[$ such that

$$\mu^{(1)}(\kappa, H) > 0.$$

Let $H_1 = \sqrt{\kappa}$ and choose κ_0 large enough, thanks to the upper bound in (5.2), such that,

$$\mu^{(1)}(\kappa, H_1) < 0, \quad \forall \kappa \geq \kappa_0.$$

As the function $H \mapsto \mu^{(1)}(\kappa, H)$ is continuous, we get by the Intermediate Value Theorem a contradiction to the definition of $H_*(\kappa)$. $\square$

Remark 5.2. *Let*

$$\alpha(\kappa) := \frac{\kappa}{H_*(\kappa)}. \tag{5.6}$$

Then, thanks to (5.2), there exist positive constants C, C', κ_0 such that,

$$C' \leq \alpha(\kappa) \leq C, \quad \forall \kappa \geq \kappa_0. \tag{5.7}$$

Moreover, thanks to (5.3), it is sufficient to calculate $\lim_{\kappa \to +\infty} \alpha(\kappa)$ in order to prove Theorem 1.1.

We modify our notation slightly by redefining the parameter ε,

$$\varepsilon = \frac{1}{\sqrt{\kappa H_*(\kappa)}}. \tag{5.8}$$

Proposition 5.3. *(Lower bound)*
There exist constants $C, \varepsilon_0 > 0$ such that,

$$\inf_{\xi \in \mathbb{R}} \mu_1 (a, m, \alpha(\kappa); \xi) + C\varepsilon \geq 0, \quad \forall \varepsilon \in]0, \varepsilon_0], \tag{5.9}$$

where the function $\mu_1(\cdot)$ is defined in (3.5).

Proof. If $m \leq 1$, then, thanks to Theorem 3.5,

$$\inf_{\xi \in \mathbb{R}} \mu_1 (a, m, \alpha(\kappa); \xi) = 1 - \alpha(\kappa).$$

In this case, (5.9) comes from the upper bound in (5.2).
Let us suppose now that $m > 1$. If the function

$$\xi \mapsto \mu_1 \left(a, \frac{m}{\mu}, \alpha(\kappa); \xi \right)$$

does not attain its minimum, we have then nothing to prove, thanks to (3.9) and (5.2).
Suppose now that the function $\xi \mapsto \mu_1 (a, m, \alpha(\kappa); \xi)$ attains its minimum at a point $\eta = \eta (a, m, \alpha(\kappa))$. Notice that $\eta(\kappa)$ is bounded, thanks to (3.18) and (5.7). Let x_0 be an arbitrary point of $\partial\Omega$. Using the boundary coordinates (s,t) introduced in Subsection 2.2, we construct a trial function $u(s,t)$ supported near x_0. We can suppose that $x_0 = (0,0)$ in the (s,t) coordinate system. Let χ be a cut-off function such that,

$$\operatorname{supp} \chi \subset]-t_0, t_0[, \quad 0 \leq \chi \leq 1, \quad \chi = 1 \text{ in } [-t_0/2, t_0/2], \tag{5.10}$$

where the constant t_0 is the geometric constant introduced in Subsection 2.2. Let us consider another function f such that,

$$f \in C_0^\infty(]-1/2, 1/2[), \quad \|f\|_{L^2(\mathbb{R})} = 1. \tag{5.11}$$

We denote also by $f_{\alpha,\eta}$ the first (positive) eigenfunction of the operator $H[a, \frac{m}{\mu}, \alpha(\kappa); \eta]$ whose L^2-norm in $\mathbb{R}$ is equal to 1.
We define $u(s,t)$ by,

$$u(s,t) = a^{-1/2}(s,t)\varepsilon^{-3/4} \exp\left(-\frac{i\eta s}{\varepsilon}\right) f_{\alpha,\eta}\left(\varepsilon^{-1}t\right) \chi(t) \times f(\varepsilon^{-1/2}s). \tag{5.12}$$

We recall the definition of $a(s,t)$ in (2.19). Since the function $f_{\alpha,\eta}$ decays exponentially at infinity, we get, thanks to (2.23),

$$1 - C\exp\left(-\frac{1}{\varepsilon}\right) \leq \|u\|^2_{L^2(\mathbb{R}^2)} \leq 1. \tag{5.13}$$

Working with the gauge introduced in Proposition 2.3, we get, thanks to the change of variable formulas (2.21) and (2.22) :

$$\left|\varepsilon^2 \mathcal{Q}[\kappa, H_*](u) - Q\left[a, \frac{m}{\mu}, \alpha(\kappa); \eta\right] (f_{\alpha,\eta})\right| \leq C\varepsilon. \tag{5.14}$$

By our choice of $f_{\alpha,\eta}$ and since $\mu_1(\kappa, H_*) = 0$, the application of the min-max principle achieves the proof. □

Proposition 5.4. *(Upper bound)*
There exist constants $C, \varepsilon_0 > 0$ such that, if $\varepsilon \in]0, \varepsilon_0]$, then,

$$\left(1 - C\varepsilon^{1/2}\right) \inf_{\xi\in\mathbb{R}} \mu_1\left(a, m, \alpha(\kappa); \xi\right) - C\varepsilon^{1/2} \leq 0, \tag{5.15}$$

where ε has been introduced in (5.8).

Proof. We follow the technique of Helffer-Morame [24, Subsection 6.3] and we localize by means of a partition of unity to compare with the model operator.
Let $0 < \rho < 2$. Consider the partition of unity (χ_j^ε) defined by Proposition 2.6. We have the following decomposition formula, thanks to (2.35),

$$Q[\kappa, H_*](u) = \sum_j Q[\kappa, H_*](\chi_j^\varepsilon u) - \sum_j \| \, |\nabla \chi_j^\varepsilon| \, u\|^2, \quad \forall u \in \mathcal{H}^1_{\varepsilon^{-2}\mathbf{F}}(\mathbb{R}^2), \tag{5.16}$$

where $\|.\|$ denotes the L^2-norm in $\mathbb{R}^2$.
The alternative appearing in (2.34) permits one to decompose the above sum in the following form :

$$\sum = \sum_{\text{int}} + \sum_{\text{ext}} + \sum_{\text{bnd}},$$

where the summation over 'int' means that we sum over the j's such that χ_j^ε is supported in Ω, that over 'ext' means that χ_j^ε is supported in $\overline{\Omega}^c$ and that over 'bnd' means that the support of χ_j^ε meets the boundary $\partial\Omega$.
We have to bound from below each of the terms on the right hand side of (5.16). By (2.33), we can estimate the contribution of the last term in (5.16) :

$$\sum_j \| \, |\nabla \chi_j^\varepsilon| \, u\|^2 \leq C\zeta_0^{-2}\varepsilon^{-2\rho}\|u\|^2. \tag{5.17}$$

Using (2.30) and our choice of $\mathbf{F}$ in (**??**), we get,

$$\sum_{\text{int}} \mathcal{Q}[\kappa, H_*](\chi_j^\varepsilon u) \geq \varepsilon^{-2} \sum_{\text{int}} \left(1 - \frac{\kappa}{H_*}\right) \|\chi_j^\varepsilon u\|^2, \tag{5.18}$$

$$\sum_{\text{ext}} \mathcal{Q}[\kappa, H_*](\chi_j^\varepsilon u) \geq \varepsilon^{-2} \sum_{\text{ext}} \left(\frac{1}{m} + a\frac{\kappa}{H_*}\right) \|\chi_j^\varepsilon u\|^2. \tag{5.19}$$

We have only now to bound from below $\sum_{\text{bnd}} \mathcal{Q}[\kappa, H_*](\chi_j^\varepsilon u)$. In the support of χ_j^ε, we choose the gauge from Proposition 2.3. Proposition 2.5 yields now the existence of constants $C, \varepsilon_0 > 0$ depending only on Ω, and a function $\phi_j \in H^1_{\text{loc}}(\mathbb{R}^2)$ such that, for any $\theta > 0$ and $\varepsilon \in]0, \varepsilon_0]$,

$$\begin{aligned}\sum_{\text{bnd}} \mathcal{Q}[\kappa, H_*](\chi_j^\varepsilon u) \geq \left(1 - C\zeta_0\varepsilon^\rho - C\varepsilon^{2\theta}\right) \sum_{\text{bnd}} \mathcal{Q}_{\mathbb{R}\times\mathbb{R}_+}[\kappa, H_*]\left(\exp\left(-i\frac{\phi_j}{\varepsilon}\right)\chi_j^\varepsilon u\right)\\ -C\varepsilon^{-2}\left(\varepsilon^{4\rho-2\theta-2} + \zeta_0\varepsilon^\rho + \varepsilon^{2\theta}\right)\sum_{\text{brd}} \|\chi_j^\varepsilon u\|^2,\end{aligned} \tag{5.20}$$

where the quadratic form $\mathcal{Q}_{\mathbb{R}\times\mathbb{R}_+}[\kappa, H_*]$ is defined by (1.10). Notice that we have also used the estimate (5.7). Formula (5.20) now reads, thanks to Remark 3.2,

$$\begin{aligned}\sum_{\text{bnd}} Q[\kappa, H_*](\chi_j^\varepsilon u) \geq \varepsilon^{-2}\left(1 - C\zeta_0\varepsilon^\rho - C\varepsilon^{2\theta}\right)\left(\inf_{\xi\in\mathbb{R}} \mu_1(a, m, \alpha(\kappa); \xi)\right)\sum_{\text{bnd}} \|\chi_j^\varepsilon u\|^2\\ -C\varepsilon^{-2}\left(\varepsilon^{4\rho-2\theta-2} + \zeta_0\varepsilon^\rho + \varepsilon^{2\theta}\right)\sum_{\text{bnd}} \|\chi_j^\varepsilon u\|^2.\end{aligned} \tag{5.21}$$

Summing up the estimates (5.17), (5.18), (5.19) and (5.21), the decomposition formulas (5.16) read as,

$$\begin{aligned}\mathcal{Q}[\kappa, H_*](\chi_j^\varepsilon u) \quad \geq \quad & \varepsilon^{-2}\left(\sum_{\text{int}}\left(1 - \frac{\kappa}{H_*}\right)\|\chi_j^\varepsilon u\|^2 + \sum_{\text{ext}}\left(\frac{1}{m} + a\frac{\kappa}{H_*}\right)\|\chi_j^\varepsilon u\|^2\right)\\ & +\varepsilon^{-2}\left(1 - C\zeta_0\varepsilon^\rho - C\varepsilon^{2\theta}\right)\left(\inf_{\xi\in\mathbb{R}} \mu_1(a, m, \alpha(\kappa); \xi)\right)\sum_{\text{bnd}} \|\chi_j^\varepsilon u\|^2\\ & -C\varepsilon^{-2}\left(\varepsilon^{4\rho-2\theta-2} + \zeta_0\varepsilon^\rho + \varepsilon^{2\theta} + \varepsilon^{2-2\rho}\right)\|u\|^2.\end{aligned} \tag{5.22}$$

The optimal choice of ρ and θ seems to be when $4\rho - 2\theta - 2 = 2 - 2\rho = 2\theta$, i.e. $\rho = 3/4$ and $\theta = 1/4$. With this choice, and taking $\zeta_0 = 1$, we get the lower bound (5.15) after the application of the min-max principle and by remembering that $\mu_1(\kappa, H_*) = 0$. Notice that we have used also the following fact which results from (3.9),

$$\inf_{\xi\in\mathbb{R}} \mu_1(a, m, \alpha(\kappa); \xi) \leq \min\left(1 - \alpha(\kappa), \frac{1}{m} + \alpha(\kappa)\right).$$

□

Proof of Theorem 1.1. Let κ_n be a sequence such that :

$$\lim_{n\to+\infty}\kappa_n=+\infty,\quad \exists\,\alpha_0>0\ :\ \lim_{n\to+\infty}\alpha(\kappa_n)=\alpha_0.$$

Then, thanks to Propositions 5.3, 5.4 and the regularity in Theorem 3.1, we get,

$$\inf_{\xi\in\mathbb{R}}\mu_1\left(a,m,\alpha_0;\xi\right)=0.$$

Therefore, by Theorem 3.7, we should have,

$$\alpha_0=\alpha\left(a,m\right).$$

This achieves the proof of Theorem 1.1, thanks to Remark 5.2. □

6 Existence and decay of eigenfunctions

Let us denote by $\mathcal{P}$ the self-adjoint operator associated to the quadratic form (1.5) when $\kappa=H=1$. The bottom of the essential spectrum for a Schrödinger operator with electric potential is characterized by Persson's Lemma [35]. The proof of Persson's Lemma in [1] can be imitated so that we obtain the following characterization of the essential spectrum of $\mathcal{P}$ (see [29, Theorem 9.4.1]).

Lemma 6.1. *Suppose that $\Omega\subset\mathbb{R}^2$ is bounded. The bottom of the essential spectrum of $\mathcal{P}$ is given by,*

$$\inf\sigma_{\rm ess}(\mathcal{P})=\Sigma(\mathcal{P}),$$

where,

$$\Sigma(\mathcal{P})=\sup_{\mathcal{K}\subset\Omega^c}\inf\left\{\frac{1}{m}\|\nabla_{\mathbf{F}}\phi\|^2_{L^2(\mathbb{R}^2)}+a;\quad \phi\in C_0^\infty(\Omega^c\setminus\mathcal{K}),\quad \|\phi\|_{L^2(\mathbb{R}^2)}=1\right\},$$

and the upper bound above is taken over all compact sets $\mathcal{K}$ in Ω^c.

Using the above characterization of the essential spectrum, one can obtain the existence of eigenfunctions of the operator $\mathcal{P}[\kappa,H]$.

Proposition 6.2. *Let $H=H_{C_3}(a,m;\kappa)$. There exists a constant $\kappa_0>0$ such that, if $\kappa\geq\kappa_0$, then $\mu^{(1)}(\kappa,H)$ is an eigenvalue of the operator $\mathcal{P}[\kappa,H]$.*
Moreover, if $m>1$, denoting by ϕ_κ a ground state of $\mathcal{P}[\kappa,H]$, then ϕ_κ is exponentially localized near the boundary in the following sense,

$$\begin{aligned}
&\exists\,\varepsilon_0,\delta\in]0,1],\quad \exists\,C>0,\qquad \forall\,\varepsilon\in]0,\varepsilon_0],\\
&\left\|\exp\left(\delta\frac{\mathrm{dist}(x,\partial\Omega)}{\varepsilon}\right)\phi_\kappa\right\|_{L^2(\mathbb{R}^2)}\leq C\|\phi_\kappa\|^2_{L^2(\mathbb{R}^2)},\\
&\left\|\exp\left(\delta\frac{\mathrm{dist}(x,\partial\Omega)}{\varepsilon}\right)\phi_\kappa\right\|_{H^1(\mathbb{R}^2)}\leq C\varepsilon^{-1}\|\phi_\kappa\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}\tag{6.1}$$

Proof. Using Lemma 6.1, we get, thanks to (2.30),

$$\inf \sigma_{\mathrm{ess}}\left(\mathcal{P}[\kappa, H]\right) \geq \kappa H\left(a\frac{\kappa}{H}+\frac{1}{m}\right).$$

Proposition 5.3 gives on the other hand,

$$\mu^{(1)}(\kappa, H) \leq (\kappa H)\epsilon(\kappa),$$

where

$$\epsilon(\kappa) := \inf_{\xi\in\mathbb{R}} \mu_1\left(a, m, \alpha(\kappa); \xi\right), \tag{6.2}$$

and $\lim\limits_{\kappa\to+\infty} \epsilon(\kappa) = 0$. Therefore, there exist $\kappa_0 > 0$ such that, if $\kappa \geq \kappa_0$, then,

$$\mu^{(1)}(\kappa, H) < \inf \sigma_{\mathrm{ess}}(\mathcal{P}[\kappa, H]),$$

and therefore $\mu^{(1)}(\kappa, H)$ is an eigenvalue.
We obtain the localization of the ground states via Agmon's technique (cf. [1, 24]). Let us explain briefly the argument. Let Φ be a Lipschitz function with compact support. An integration by parts yields the following identity,

$$\begin{aligned}\mathcal{Q}[\kappa, H]\left(\exp\left(\frac{\Phi}{\varepsilon}\right)\phi_\kappa\right) &= \mu^{(1)}(\kappa, H)\left\|\exp\left(\frac{\Phi}{\varepsilon}\right)\phi_\kappa\right\|^2_{L^2(\mathbb{R}^2)} \\ &+\varepsilon^{-2}\left\||\nabla\Phi|\exp\left(\frac{\Phi}{\varepsilon}\right)\phi_\kappa\right\|^2_{L^2(\Omega)} + \frac{\varepsilon^{-2}}{m}\left\||\nabla\Phi|\exp\left(\frac{\Phi}{\varepsilon}\right)\phi_\kappa\right\|^2_{L^2(\Omega^c)}.\end{aligned} \tag{6.3}$$

We denote by,

$$u = \exp\left(\frac{\Phi}{\varepsilon}\right)\phi_\kappa, \quad \beta = \min\left(1-\frac{\kappa}{H}, \frac{1}{m}+a\frac{\kappa}{H}\right), \quad \gamma = \max\left(1, \frac{1}{m}\right).$$

Using the lower bound for $\mathcal{Q}[\kappa, H]$ in (5.22) (with $\rho = 1$ and $\theta = 1/2$) and the upper bound for $\mu^{(1)}(\kappa, H)$ in (5.9), we rewrite (6.3) as follows,

$$\begin{aligned}&\left(\beta - \epsilon(\kappa) - C\zeta_0 - \gamma\|\nabla\Phi\|^2_{L^\infty(\mathbb{R}^2)} - C\varepsilon^{1/2}\right)\sum_{\mathrm{int,ext}}\|\chi_j^\varepsilon u\|^2 \\ &\leq \left(\gamma\|\nabla\Phi\|^2_{L^\infty(\mathbb{R}^2)} + C\zeta_0^{-2} + C\varepsilon^{1/2}\right)\sum_{\mathrm{bnd}}\|\chi_j^\varepsilon u\|^2,\end{aligned} \tag{6.4}$$

where, thanks to the choice of ρ, each χ_j^ε is supported in a disk of radius $\zeta_0\varepsilon$.
Given an integer N, we choose

$$\Phi = \delta\chi\left(\frac{|x|}{N}\right)\Phi_0(x),$$

where $\delta \in]0, 1]$ is to be determined, χ is a cut-off function,

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ in } \left[0, \frac{1}{2}\right], \quad \mathrm{supp}\chi \subset [0, 1],$$

and Φ_0 is defined by

$$\Phi_0(x) = \max\left(\mathrm{dist}(x, \partial\Omega), \varepsilon\right).$$

Since $m > 1$, we can choose ζ_0 small enough and κ_0 large enough such that,

$$\beta - \epsilon(\kappa) - C\zeta_0 > \frac{\beta}{2}, \quad \forall \kappa \geq \kappa_0.$$

There exist also $N_0, \delta_0 > 0$ such that, for $N \geq N_0$ and $\delta \in]0, \delta_0]$,

$$\frac{\beta}{2} - \gamma \|\nabla \Phi\|^2_{L^\infty(\mathbb{R}^2)} \geq \frac{\beta}{4}.$$

Therefor, we can rewrite (6.4) as,

$$\int_{\mathbb{R}^2, |x| \leq N} \left| \exp\left(\frac{\delta \mathrm{dist}(x, \partial\Omega)}{\varepsilon} \right) \phi_\kappa \right| dx \leq \widetilde{C} \|\phi_\kappa\|^2,$$

for a constant $\widetilde{C} > 0$. Noticing that the above estimate is uniform with respect to $N \geq N_0$, we get (6.1) by passing to the limit $N \to +\infty$. The bound on the H^1-norm follows now from (6.3). $\square$

As a result of the decay in (6.1), we get another localization version of the ground states.

Lemma 6.3. *If $m > 1$, then, given an integer $k \in \mathbb{N}$, there exist constants $\varepsilon_k, C_k > 0$ such that, for any ground state ϕ_κ and $\varepsilon \in]0, \varepsilon_k]$, we have,*

$$\int_{\mathbb{R}^2} |t(z)|^k |\phi_\kappa(z)|^2 dz \leq C_k \varepsilon^k, \tag{6.5}$$

$$\int_{\mathbb{R}^2} |t(z)|^k |(\nabla - i\varepsilon^{-2}\mathbf{F})\phi_\kappa(z)|^2 dz \leq C_k \varepsilon^{k-2}. \tag{6.6}$$

Remark 6.4. *It would be interesting to analyze the localization of the ground states when $m \leq 1$. It seems in this case that the ground states should be localized in a compact subset of Ω, as far as possible of $\partial\Omega$.*

7 Curvature effects and proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. The computations that we shall carry are similar to those carried out in [24, Sections 10-11].
Given $a > 0$, we shall suppose that,

$$m \geq m_0, \tag{7.1}$$

where $m_0 > 1$ is defined in Theorem 3.17. We denote by,

$$\alpha = \alpha\,(a, m)\,, \quad \widehat{\alpha} = \frac{\kappa}{H}, \quad H = H_*(\kappa), \tag{7.2}$$

where $H_*(\kappa)$ is defined in (5.5) and is equal to the upper critical field for large values of κ. We shall keep the same notations introduced in Section 4. In all the proofs, C, ε_0 will denote generic constants independent of ε and that may change from line to line.

Proposition 7.1. *With the above hypotheses and notations, there exist constants $C, \varepsilon > 0$, such that, if $\varepsilon \in]0, \varepsilon_0]$, then,*

$$\varepsilon^2 \mu^{(1)}(\kappa, H) \leq \mu_1\,(a, m, \widehat{\alpha}; \widehat{\eta}) + \widehat{\mathcal{C}}_1\,(a, m)\,(\kappa_{\mathrm{r}})_{\max}\,\varepsilon + C\varepsilon^{3/2}, \tag{7.3}$$

where the function $\widehat{\mathcal{C}}_1(\cdot, \cdot)$ is defined in (4.41).

Proof. We construct a trial function inspired by [7, 24]. Let $z_0 \in \partial\Omega$ be such that $\kappa_{\rm r}(z_0) = (\kappa_{\rm r})_{\max}$, and suppose that $z_0 = 0$ in the (s,t) coordinate system introduced in Subsection 2.2. Define the following quasi-mode,

$$u_\varepsilon(s,t) = \varepsilon^{-5/8} a_0^{-1/2}(t) f_{\widehat{\alpha}}(\varepsilon^{-1}t)\chi(t) \times f(\varepsilon^{-1/4}s), \tag{7.4}$$

where $f_{\widehat{\alpha}}$ is the first eigenfunction of the operator $H[a,m,\widehat{\alpha};\widehat{\eta}]$ whose L^2-norm in $\mathbb{R}$ is equal to 1. The functions χ and f are as in (5.10) and (5.11), and $a_0(t)$ is defined by :

$$a_0(t) = 1 - t\kappa_0, \qquad \text{where } \kappa_0 := (\kappa_{\rm r})_{\max}.$$

Let us introduce the operator,

$$H_1 = \partial_t + (t-\widehat{\eta})^3 - \xi^2(t-\widehat{\eta}).$$

Then, using the exponential decay of $f_{\widehat{\alpha}}$ and the choice of gauge in Proposition 2.3, one gets (for the detailed calculations, see [29, Proposition 5.4.3]),

$$\left| q_{\varepsilon,\mathbf{F},\Omega}(u_\varepsilon) - \varepsilon^{-2}\int_{\mathbb{R}_+} \big(|f_{\widehat{\alpha}}'(t)|^2 + |(t-\widehat{\eta})f_{\widehat{\alpha}}(t)|^2 + \kappa_0\varepsilon (H_1 f_{\widehat{\alpha}})(t)\, f_{\widehat{\alpha}}(t)\big)\,dt \right| \le C\varepsilon^{-1/2}, \tag{7.5}$$

$$\left| q_{\varepsilon,\mathbf{F},\overline{\Omega}^c}(u_\varepsilon) - \varepsilon^{-2}\int_{\mathbb{R}_-} \big(|f_{\widehat{\alpha}}'(t)|^2 + |(t-\widehat{\eta})f_{\widehat{\alpha}}(t)|^2 + \kappa_0\varepsilon (H_1 f_{\widehat{\alpha}})(t)\, f_{\widehat{\alpha}}(t)\big)\,dt \right| \le C\varepsilon^{-1/2}. \tag{7.6}$$

An integration by parts yields, thanks to (3.21) and (4.41),

$$\int_{\mathbb{R}_+} H_1\, f_{\widehat{\alpha}}(t)\, f_{\widehat{\alpha}}(t)\,dt + \frac{1}{m}\int_{\mathbb{R}_-} H_1\, f_{\widehat{\alpha}}(t)\, f_{\widehat{\alpha}}(t)\,dt = \widehat{\mathcal{C}}_1\,(a,m)\,.$$

Therefore, (7.5) and (7.6) together with our choice of $f_{\widehat{\alpha}}$ yield the estimate,

$$\left| \mathcal{Q}[\kappa,H](u_\varepsilon) - \varepsilon^{-2}\left\{ \mu_1\,(a,m,\widehat{\alpha};\widehat{\eta}) + \widehat{\mathcal{C}}_1\,(a,m)\,\kappa_0\varepsilon \right\} \right| \le C\varepsilon^{-1/2}.$$

The decay of $f_{\widehat{\alpha}}$ at infinity also yields,

$$\left| \|u_\varepsilon\|^2_{L^2(\mathbb{R})} - 1 \right| \le C\exp\left(-\varepsilon^{-1}\right), \quad \forall \varepsilon \in]0,\varepsilon_0].$$

The application of the min-max principle now achieves the proof of the proposition. □

Remark 7.2. *Using Theorem 4.4, we get a better version of the upper bound (7.3),*

$$\begin{aligned} \varepsilon^2\mu^{(1)}(\kappa,H) \ &\le\ b_1\,(a,m)\,(\widehat{\alpha}-\alpha) + \widetilde{\mathcal{C}}_1\,(a,m)\,(\kappa_{\rm r})_{\max}\,\varepsilon \\ &\quad + C\left(\varepsilon^{3/2} + |\widehat{\alpha}-\alpha|^2\right), \quad \forall\varepsilon\in]0,\varepsilon_0], \end{aligned} \tag{7.7}$$

where $b_1(\cdot,\cdot)$ and $\widetilde{\mathcal{C}}_1(\cdot,\cdot)$ are defined in (4.10) and (4.11) respectively.

In the next proposition, using the existence of ground states (cf. Proposition 6.2), we shall determine a lower bound for $\mu^{(1)}(\kappa,H)$.

Proposition 7.3. *Under the above hypotheses and notations, there exist constants $C,\varepsilon_0 > 0$, such that, if $\varepsilon\in]0,\varepsilon_0]$, then,*

$$\varepsilon^2\mu^{(1)}(\kappa,H) \ge b_1\,(a,m)\,(\widehat{\alpha}-\alpha) + \widetilde{\mathcal{C}}_1\,(a,m)\,(\kappa_{\rm r})_{\max}\,\varepsilon - C(\varepsilon^{4/3} + |\widehat{\alpha}-\alpha|^2). \tag{7.8}$$

Proof. Let us consider a partition of unity $\left(\chi_{j,\varepsilon^{1/3}}\right)_{j\in\mathbb{Z}^2}$ of $\mathbb{R}^2$ that satisfies :

$$\sum_{j\in J}|\chi_{j,\varepsilon^{1/3}}(z)|^2 = 1, \quad \sum_{j\in J}|\nabla\chi_{j,\varepsilon^{1/3}}(z)|^2 \leq C\varepsilon^{-2/3}, \tag{7.9}$$

$$\text{supp } \chi_{j,\varepsilon^{1/3}} \subset j\varepsilon^{1/3} + [-\varepsilon^{1/3}, \varepsilon^{1/3}]^2. \tag{7.10}$$

We define the following set of indices :

$$J^1_{\tau(\varepsilon)} := \{j \in \mathbb{Z}^2; \quad \text{supp}\chi_{j,\varepsilon^{1/3}} \cap \Omega \neq \emptyset, \quad \text{dist}(\text{supp } \chi_{j,\varepsilon^{1/3}}, \partial\Omega) \leq \tau(\varepsilon)\},$$

where the number $\tau(\varepsilon)$ is defined by :

$$\tau(\varepsilon) = \varepsilon^{2\delta}, \quad \text{with} \quad \frac{1}{6} \leq \delta \leq \frac{1}{2}, \tag{7.11}$$

and the number δ will be chosen suitably.
We consider also another scaled partition of unity in $\mathbb{R}$:

$$\psi^2_{0,\tau(\varepsilon)}(t) + \psi^2_{1,\tau(\varepsilon)}(t) = 1, \quad |\psi'_{j,\tau(\varepsilon)}(t)| \leq \frac{C}{\tau(\varepsilon)}, \quad j = 0, 1, \tag{7.12}$$

$$\text{supp}\,\psi_{0,\tau(\varepsilon)} \subset \left[\frac{\tau(\varepsilon)}{20}, +\infty\right[, \quad \text{supp}\,\psi_{1,\tau(\varepsilon)} \subset \left]-\infty, \frac{\tau(\varepsilon)}{10}\right]. \tag{7.13}$$

Notice that, for each $j \in J^1_{\tau(\varepsilon)}$, the function $\psi_{1,\tau(\varepsilon)}(t)\chi_{j,\varepsilon^{1/3}}(s,t)$ can be interpreted, by means of boundary coordinates, as a function in $\mathcal{N}_{t_0}$ (cf. (2.15)). Moreover, each $\psi_{1,\tau(\varepsilon)}(t)\chi_{j,\varepsilon^{1/3}}(s,t)$ is supported in a rectangle

$$K(j,\varepsilon) =]-\varepsilon^{1/3} + s_j, s_j + \varepsilon^{1/3}[\times[0, \varepsilon^{2\delta}[$$

near $\partial\Omega$. The role of δ is then to control the width of each rectangle $K(j,\varepsilon)$.
Let ϕ_κ be an L^2-normalized ground state of $\mathcal{P}[\kappa, H]$ whose existence was shown in Proposition 6.2. Since ϕ_κ decays exponentially away from the boundary, we get,

$$\left|\sum_{j\in J^1_{\tau(\varepsilon)}} \mathcal{Q}[\kappa, H](\psi_{1,\tau(\varepsilon)}\chi_{j,\varepsilon^{1/3}}\phi_\kappa) - \varepsilon^{-4}\mu^{(1)}(\kappa, H)\right| \leq C\varepsilon^{-2/3}. \tag{7.14}$$

The proof of (7.14) follows actually that of [24, Formulas (10.4), (10.5), (10,6)], see [29, Proposition 5.5.1].
For each $j \in J^1_{\tau(\varepsilon)}$, we define a unique point $z_j \in \partial\Omega$ by the relation $s(z_j) = s_j$. We denote then by $\kappa_j = \kappa_{\mathrm{r}}(z_j)$, $a_j(t) = 1 - \kappa_j t$, and

$$A^j(t) = -t\left(1 - \frac{t}{2}\kappa_j\right),$$

Let us consider the k-family of differential operators,

$$H_{\varepsilon,j,k} = -\varepsilon^4 a_j^{-1}\partial_t^2(a_j\partial_t) + (1 + 2\kappa_j t)(\varepsilon^2 k - A^j)^2.$$

We denote by $H^{a,m}_{\varepsilon,j,k}$ the self-adjoint operator on the space

$$L^2(]-\varepsilon^{2\delta}, \varepsilon^{2\delta}[; a_j(t)dt)$$

defined by :

$$H^{a,m,\widehat{\alpha}}_{\varepsilon,j,k} = \begin{cases} H_{\varepsilon,j,k} - \widehat{\alpha}, & t > 0, \\ \frac{1}{m} H_{\varepsilon,j,k} + a\widehat{\alpha}, & t < 0, \end{cases}$$

with domain :

$$\begin{aligned} D(H^{a,m,\widehat{\alpha}}_{\varepsilon,j,k}) &= \{u \in H^1_0(]-\varepsilon^{2\delta}, \varepsilon^{2\delta}[; \quad u_{|_{]-\varepsilon^{2\delta},0[}} \in H^2(]-\varepsilon^{2\delta}, 0[), \\ & \quad u_{|_{]0,\varepsilon^{2\delta}[}} \in H^2(]0, \varepsilon^{2\delta}[), \quad u'(0_+) = \frac{1}{m} u'(0_-)\}. \end{aligned}$$

We introduce :

$$\mu_1(H^{a,m,\widehat{\alpha}}_{\varepsilon,j,k}) := \inf_{k\in\mathbb{R}} \inf \mathrm{Sp}(H^{a,m,\widehat{\alpha}}_{\varepsilon,j,k}). \tag{7.15}$$

Then one gets from (7.14) the following result,

$$\varepsilon^2 \mu^{(1)}(\kappa, H) \geq \left(\inf_{j\in J^1_{\tau(\varepsilon)}} \mu_1(H^{a,m,\widehat{\alpha}}_{\varepsilon,j,k}) \right) + \mathcal{O}(\varepsilon^{4/3}). \tag{7.16}$$

Let us explain briefly how we get (7.16) (the details of the calculations are given in [24, Section 11]). We express each term $\mathcal{Q}[\kappa, H](\psi_{1,\tau(\varepsilon)} \chi_{j,\varepsilon^{1/3}} \phi_\kappa)$ in boundary coordinates. We work with the local choice of gauge given in Proposition 2.3. We expand now all the terms by Taylor's Formula near $(s_j, 0)$. After controlling the remainder terms, thanks to the exponential decay of the ground states away from the boundary, we apply a partial Fourier transformation in the tangential variable s and we get finally the desired result (see [29, Proposition 5.5.3]). Putting,

$$\beta = \kappa_j, \quad \xi = -\varepsilon k, \quad h = \varepsilon^2,$$

and applying the scaling $\widetilde{t} = h^{-1/2} t$, one gets,

$$\mu_1(H^{a,m,\widehat{\alpha}}_{\varepsilon,j,k}) = h \mu_1(\mathcal{H}^{a,m,\widehat{\alpha}}_{h,\beta,\xi}).$$

By applying Theorem 4.9[7], we finish the proof of the theorem. □

Proof of Theorem 1.3. It results from (7.7), (7.8) and the definition of $H_*(\kappa)$,

$$b_1(a,m)(\widehat{\alpha} - \alpha) + \widetilde{\mathcal{C}}_1(a,m)(\kappa_{\mathrm{r}})_{\max} \varepsilon + \mathcal{O}\left(\varepsilon^{4/3} + |\widehat{\alpha} - \alpha|^2\right) = 0.$$

This yields,

$$|\widehat{\alpha} - \alpha| \leq C\varepsilon,$$

and consequently,

$$H_*(\kappa) = \frac{\kappa}{\alpha} - \frac{\widetilde{\mathcal{C}}_1(a,m)}{b_1(a,m)\,\alpha^{3/2}} (\kappa_{\mathrm{r}})_{\max} + \mathcal{O}(\kappa^{-1/3}).$$

This achieves the proof of the theorem upon setting,

$$\mathcal{C}_1(a,\cdot) = -\frac{\widetilde{\mathcal{C}}_1(a,\cdot)}{b_1(a,\cdot)}. \tag{7.17}$$

[7] The optimal choice of δ which gives a remainder in accordance with (7.16) is $\delta = 5/12$. We have to notice also that $\widetilde{C}_1(a,m)$ is negative for $m \geq m_0$.

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A Proof of Theorem 1.6

We bring in this appendix the proof of the asymptotics (1.22). Notice that the Euler-Lagrange equations associated to the functional (1.20) has a solution $(0, \mathbf{A})$ that satisfies,

$$\operatorname{curl} \mathbf{A} = 1, \quad \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega, \quad \nu \cdot \mathbf{A} = 0 \text{ on } \partial\Omega,$$

and the normal state $(0, \mathbf{A})$ is unique. Define the following eigenvalue,

$$\lambda^{(1)}(\delta, \gamma_0; \varepsilon) = \inf_{u \in H^1(\Omega), u \neq 0} \frac{q_\varepsilon^{\delta,\gamma_0}(u)}{\|u\|^2_{L^2(\Omega)}}, \tag{A.1}$$

where,

$$q_\varepsilon^{\delta,\gamma_0}(u) = \|(\nabla - i\varepsilon^{-2}\mathbf{A})u\|^2_{L^2(\Omega)} + \kappa^\delta \gamma_0 \|u\|^2_{L^2(\partial\Omega)},$$

and $\varepsilon = \frac{1}{\sqrt{\kappa H}}$. We define the following upper critical field,

$$H_{C_3}(\delta, \gamma_0; \kappa) = \inf\{H > 0; \quad \lambda^{(1)}(\delta, \gamma_0; \varepsilon) \geq \kappa^2\}.$$

As for (5.3), we can also show that the critical field,

$$H_*(\delta, \gamma_0; \kappa) = \inf\{H > 0; \quad \lambda^{(1)}(\delta, \gamma_0; \varepsilon) = \kappa^2\}, \tag{A.2}$$

is equal to the upper critical field for large values of κ. Following the generalization of the analysis of Helffer-Morame [24] in [30], we obtain the following asymptotics for $\lambda^{(1)}(\delta, \gamma_0; \varepsilon)$.

Proposition A.1. *Let $H = H_*(\kappa, \delta, \gamma_0)$. The following asymptotics holds as ε tends* 0*,*

$$\varepsilon^2 \lambda^{(1)}(\delta, \gamma_0; \varepsilon) = \Theta\left(\varepsilon^2 \kappa^\delta \gamma_0\right)(1 + o(1)), \tag{A.3}$$

where $\Theta(\cdot)$ is defined in (2.8).

Proof. We split the proof in two steps corresponding to the determination of an upper bound and of a lower bound.
Step 1. Upper bound.
We establish the following upper bound,

$$\varepsilon^2 \lambda^{(1)}(\delta, \gamma_0; \varepsilon) \leq \Theta\left(\varepsilon^2 \kappa^\delta \gamma_0\right)(1 + o(1)) \quad (\varepsilon \to 0). \tag{A.4}$$

We have two cases to deal with,

$$\text{either } \lim_{\varepsilon \to 0} \varepsilon^2 \kappa^\delta = +\infty, \text{ or } \lim_{\varepsilon \to 0} \varepsilon^2 \kappa^\delta < +\infty.$$

Case 1. $\lim_{\varepsilon\to0}\varepsilon^2\kappa^\delta=+\infty$ *and* $\gamma_0>0$. In this case, by the min-max principle and (2.27), we get,

$$\varepsilon^2\lambda^{(1)}(\delta,\gamma_0;\varepsilon)\le\varepsilon^2\mu^D(\varepsilon;\Omega)\le 1+C\varepsilon^3.$$

We obtain then the upper bound (A.4) upon recalling that $\Theta(\gamma)$ is exponentially close to 1 when $\gamma\to+\infty$ (cf. [30]).
Case 2. $\lim_{\varepsilon\to0}\varepsilon^2\kappa^2<+\infty$ *or* $\gamma_0\le0$. Let $\eta:=\varepsilon^2\kappa^\delta\gamma_0$. We construct the following quasi-mode by means of boundary coordinates (cf. Subsection 2.2),

$$u_\varepsilon(s,t)=\varepsilon^{-3/4}a^{-1/2}\exp\left(-i\frac{\xi(\eta)s}{\varepsilon}\right)\varphi_\eta\left(\varepsilon^{-1}t\right)\chi(t)\times f\left(\varepsilon^{-1/2}s\right),$$

where the functions χ and f are as in (5.10) and (5.11) respectively. Then following [30, Proof of Proposition 3.1], we get the upper bound (A.4).
Step 2. Lower bound.
We establish the following lower bound,

$$\varepsilon^2\lambda^{(1)}(\delta,\gamma_0;\varepsilon)\ge\Theta\left((1-C\varepsilon^{1/2})\varepsilon^2\kappa^\delta\right)-C\varepsilon^{5/2}. \tag{A.5}$$

We have, thanks to (2.35), (2.33), (2.30) and (2.31) (with the choice $\rho=3/4$ and $\theta=1/4$),

$$q_\varepsilon^{\delta,\gamma_0}(u)\ge\varepsilon^{-2}\left(\sum_{\text{int}}\|\chi_j^\varepsilon u\|^2+\left(1-C\varepsilon^{1/2}\right)\sum_{\text{bnd}}q^{\delta,\widetilde{\eta}}_{\varepsilon,\mathbb{R}\times\mathbb{R}_+}(\chi_j^\varepsilon u)-C\varepsilon^{1/2}\|u\|^2\right),$$

where $\widetilde{\eta}=(1-C\varepsilon)^{-1}\eta$. Applying the min-max principle, we get the lower bound (A.5). We achieve now the proof of the theorem upon recalling the asymptotic behavior of $\Theta(\cdot)$ (cf. [30]).
□

Lemma A.2. *There exists a unique $\eta_0<0$ such that $\Theta(\eta_0)=0$. Moreover, given $\gamma_0\in\mathbb{R}$, there exists a unique $\ell(\gamma_0)>0$ such that $\Theta(\gamma_0\cdot\ell(\gamma_0))=\ell(\gamma_0)^2$.*

Proof. The existence and uniqueness of η_0 comes from the monotonicity of $\Theta(\cdot)$ and the behavior of $\Theta(\cdot)$ at $-\infty$ ($\Theta(\gamma)\sim-\gamma^2$).
Let us define the function $h(\eta)=\Theta(\gamma_0\eta)-\eta^2$. Notice that $h(0)=\Theta_0>0$ and $h(\eta)<0$ for all $\eta\ge1$. Therefore, thanks to the Intermediate Value Theorem, there exists a solution $\ell(\gamma_0)$ of $h(\ell(\gamma_0))=0$ and this solution is in $]0,1[$. Using (2.11), we get,

$$h'(\eta)=\gamma_0|\varphi_{\gamma_0\eta}(0)|^2-2\eta,$$

where $h'(\eta)<0$ in $]0,1[$ if $\gamma_0\le0$. If $\gamma_0>0$, then thanks to the Min-Max Principle, (2.9) and (2.5),

$$\gamma_0\eta|\varphi_{\gamma_0\eta}(0)|^2\le\Theta(\gamma_0\eta)-\Theta_0,\quad \Theta_0<\ell(\gamma_0)<1,$$

and consequently, (recall that $\Theta_0>\frac12$),

$$h'(\eta)\le\frac{\Theta(\gamma_0\eta)-1}{\eta}-2\eta\le\frac{1-\Theta_0}{\eta}-2\eta<0\quad\text{in }]\Theta_0,1[.$$

Therefore, $\ell(\gamma_0)$ is unique. □

Proof of Theorem 1.6. Using (A.2) and Proposition A.1, we have to analyze the limit of $\frac{\kappa^\delta}{(\kappa H_*(\kappa))^{1/2}}$ as $\kappa \to +\infty$. Let

$$\beta = \lim_{\kappa \to +\infty} \frac{\kappa^\delta}{(\kappa H_*(\kappa))^{1/2}}, \tag{A.6}$$

with $0 \leq \beta \leq +\infty$. Notice that, thanks to (A.3) and to the definition of $H_*(\kappa)$, we have the following equation,

$$\Theta\left(\frac{\kappa^\delta}{(\kappa H_*(\kappa))^{1/2}}\gamma_0\right) = \frac{\kappa}{H_*(\kappa)}(1 + o(1)) \quad \text{as } \kappa \to +\infty. \tag{A.7}$$

The above equation gives, thanks to the monotonicity of $\Theta(\cdot)$ and the definition of η_0,

$$\beta\gamma_0 \geq \eta_0. \tag{A.8}$$

Case 1. $\delta < 1$.
We show in this case that $\beta = 0$. Suppose by contradiction that $\beta > 0$. If $\beta < +\infty$, then, thanks to (A.6),

$$H_*(\kappa) = \frac{1}{\beta^2}\kappa^{2\delta-1}(1 + o(1)).$$

Substituting in (A.7), we get,

$$\Theta(\beta\gamma_0) = \frac{1}{\beta^2}\kappa^{2(1-\delta)}(1 + o(1)),$$

which is impossible since $\delta < 1$. If $\beta = +\infty$, then, thanks to (A.8), this case is possible only if $\gamma_0 > 0$. Using (A.7), we get, thanks to the decay of $\Theta(\cdot)$ at $+\infty$,

$$\frac{\kappa}{H_*(\kappa)} = 1 + o(1),$$

and consequently,

$$\frac{\kappa^\delta}{(\kappa H_*(\kappa))^{1/2}} = \mathcal{O}(\kappa^{\delta-1}),$$

which is a contradiction since $\delta < 1$.
Case 2. $\delta = 1$.
If $\beta = +\infty$, we get a contradiction as in the above case. Therefore, $\beta < +\infty$. Then, combining (A.6) and (A.7), we get $\Theta(\beta\gamma_0) = \beta^2$. Thus, by Lemma A.2, $\beta = \ell(\gamma_0)$.
Case 3. $\delta > 1$ *and* $\gamma_0 > 0$.
It is sufficient to prove that $\beta = +\infty$. Suppose by contradiction that $\beta < +\infty$. Then, (A.7) will give $H_*(\kappa) = \kappa(1 + o(1))$ while (A.6) will give

$$H_*(\kappa) = \frac{1}{\beta^2}\kappa^{2\delta-1}(1 + o(1)),$$

which is impossible since $\delta > 1$.
Case 4. $\delta > 1$ *and* $\gamma_0 < 0$.
In this case β should be finite, thanks to (A.8). We deduce then from (A.6) that

$$H_*(\kappa) = \frac{1}{\beta^2}\kappa^{2\delta-1}(1 + o(1)).$$

Since $\delta > 1$, (A.7) gives : $\Theta(\beta\gamma_0) = 0$. Therefore, by Lemma A.2, $\beta\gamma_0 = \eta_0$. $\square$

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